Complex Analysis

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These are the lecture notes to my 2nd year Bachelor lecture in the summer semester 2013 on complex analysis in one variable. The manuscript differs from the lecture: It does not contain any pictures, and the lecture is in German.

Contents

1 Holomorphic maps

(A) Complex numbers and complex vector spaces

Reminder 1.1.

(1) $\mathbb{C}$ is a field, and $\mathbb{R}$ is a subfield. Let $\mathbb{C}^\times := \mathbb{C} \setminus \{0\}$.

(2) Every $z \in \mathbb{C}$ has a unique representation $z = x + iy$ with $x = \mathrm{Re}(z) \in \mathbb{R}$ the *real part of* z, $y = \mathrm{Im}(z) \in \mathbb{R}$ the *imaginary part of* z and with $i \in \mathbb{C}$ such that $i^2 = -1$. We denote by

$$|z| := \sqrt{x^2 + y^2} \in \mathbb{R}^{\geq 0}$$

the *absolute value of* z and by

$$\bar{z} := x - iy \in \mathbb{C}$$

the *complex conjugate of* z.

(3) For $z, w \in \mathbb{C}$ one has

$$\left.\begin{array}{r} |z| = 0 \Leftrightarrow z = 0, \\ |z+w| \leq |z| + |w|, \end{array}\right\} \qquad (\Rightarrow (\mathbb{C}, |\ |) \text{ is normed } \mathbb{R}\text{-vector space})$$
$$|zw| = |z||w|.$$

(4) Every $z \in \mathbb{C}^\times$ has a unique expression of the form

$$z = re^{i\varphi},$$

where $r = |z| \in \mathbb{R}^{>0}$ and $\varphi \in [0, 2\pi) = \mathbb{R}/2\pi\mathbb{Z}$ is the *argument of* z.

Remark 1.2. Let V be a finite-dimensional $\mathbb{C}$-vector space. Recall that all norms on V are equivalent and hence define the same topology. We always endow V with this topology.
If $V = \mathbb{C}^n$, then a subset $U \subseteq V$ is open if and only if for all $z = (z_1, \dots, z_n) \in U$ there exists $\varepsilon > 0$ such that

$$B_\varepsilon(z) := \{\, w \in \mathbb{C}^n \ ; \ |w_i - z_i| < \varepsilon \text{ for all } i = 1, \dots, n\} \subseteq U.$$

Remark 1.3. Let V, W be $\mathbb{C}$-vector spaces.
(1) We may V also consider as an $\mathbb{R}$-vector space with

$$\dim_{\mathbb{R}}(V) = 2 \dim_{\mathbb{C}}(V),$$

where $2 \cdot \infty := \infty$.
(2) For a map $A \colon V \to W$ we have

$$A \text{ is } \mathbb{C}\text{-linear} \Leftrightarrow A \text{ is } \mathbb{R}\text{-linear and } A(iv) = iA(v) \text{ for all } v \in V$$

We denote the $\mathbb{C}$-linear maps $V \to W$ by $\mathrm{Hom}_{\mathbb{C}}(V, W)$ and the $\mathbb{R}$-linear maps $V \to W$ by $\mathrm{Hom}_{\mathbb{R}}(V, W)$.

(B) Complex differentiable maps

Notation: Let V, V' be always finite-dimensional $\mathbb{C}$-vector spaces.

Definition 1.4. $U \subseteq V$ open, $f \colon U \to V'$ a map.
(1) Let $\tilde{z} \in U$. Then f is called *complex differentiable in* $\tilde{z}$ if there exists a $\mathbb{C}$-linear map

$$Df(\tilde{z}) \colon V \to V'$$

such that

$$\lim_{z \to \tilde{z}} \frac{f(z) - f(\tilde{z}) - Df(\tilde{z})(z - \tilde{z})}{\|z - \tilde{z}\|} = 0, \tag{1.4.1}$$

where $\| \cdot \|$ is any norm on V.

(2) f is called *holomorphic on U* if f is continuously complex differentiable on U, i.e. f is complex differentiable in all $\tilde{z} \in U$ and the map

$$U \to \operatorname{Hom}_{\mathbb{C}}(V, V'), \qquad \tilde{z} \mapsto Df(\tilde{z})$$

is continuous.

(3) We set

$$\mathscr{O}(U, V') := \{ f \colon U \to V' \; ; \; f \text{ holomorphic}\},$$
$$\mathscr{O}(U) := \mathscr{O}(U, \mathbb{C}).$$

Remark 1.5. $U \subseteq V$ open, $f \colon U \to V'$ a map. Then f is complex differentiable in $\tilde{z}$ if and only if f is differentiable in $\tilde{z}$ (in the sense of real analysis) and the $\mathbb{R}$-linear map $Df(\tilde{z}) \colon V \to V'$ is $\mathbb{C}$-linear. In particular the results of Analysis 2 show:
(1) $Df(\tilde{z})$ is uniquely determined by (1.4.1).
(2) If f is complex differentiable in $\tilde{z}$, then f is continuous in $\tilde{z}$.
(3) f is holomorphic in $U \Leftrightarrow f$ is a $\mathcal{C}^1$-map and $Df(\tilde{z})$ is $\mathbb{C}$-linear for all $\tilde{z} \in U$.

Proposition 1.6 (Chain rule). *V, V', V'' finite-dimensional $\mathbb{C}$-vector spaces, $U \subseteq V$, $U' \subseteq V'$ open. Let $f \colon U \to V'$, $g \colon U' \to V''$ holomorphic with $f(U) \subseteq U'$. Then $g \circ f \colon U \to V''$ is holomorphic and for all $\tilde{z} \in U$:*

$$D(g \circ f)(\tilde{z}) = Dg(f(\tilde{z})) \circ Df(\tilde{z}).$$

Proof. This follows immediately from 1.5 and the analogous assertion for real differentiable maps (because the composition of $\mathbb{C}$-linear maps is again $\mathbb{C}$-linear). ☐

Example 1.7.
(1) An $\mathbb{R}$-linear map $A \colon V \to V'$ is holomorphic if and only if A is $\mathbb{C}$-linear: One has $DA(\tilde{z}) = A$ for all $\tilde{z} \in V$.
(2) The $\mathbb{R}$-linear map $\mathbb{C} \to \mathbb{C}$, $z \mapsto \bar{z}$ is *not* holomorphic.
(3) The addition $V \oplus V \to V$, $(z, w) \mapsto z + w$ and the i-th projection $\mathbb{C}^n \to \mathbb{C}$, $(z_1, \dots, z_n) \mapsto z_i$ ($i \in \{1, \dots, n\}$) are $\mathbb{C}$-linear and hence holomorphic.
(4) The multiplication $\mu \colon \mathbb{C} \oplus \mathbb{C} \to \mathbb{C}$, $(z, w) \mapsto zw$ is a $\mathcal{C}^1$-map with

$$D\mu(\tilde{z}, \tilde{w}) \colon \mathbb{C}^2 \to \mathbb{C}, \qquad (u, v) \mapsto u\tilde{w} + v\tilde{z}$$

for all $(\tilde{z}, \tilde{w}) \in \mathbb{C}^2$. Hence $D\mu(\tilde{z}, \tilde{w})$ is $\mathbb{C}$-linear and μ is holomorphic.

(5) Applying the chain rule, (3) and (4) several times we see that polynomial mappings

$$\mathbb{C}^n \to \mathbb{C}, \qquad (z_1, \dots, z_n) \mapsto \sum_{i_1, \dots, i_n \in \mathbb{N}_0} a_{i_1 \dots i_n} z_1^{i_1} \dots z_n^{i_n}$$

(where $a_{i_1 \dots i_n} \in \mathbb{C}$ is zero for all but finitely many $i_1, \dots, i_n$) are always holomorphic.

Corollary 1.8. *Let $U \subseteq V$ open.*

(1) *$\mathscr{O}(U,V')$ is a $\mathbb{C}$-subvectorspace of the space of all maps $U \to V'$. For $f,g \in \mathscr{O}(U,V')$, $\alpha \in \mathbb{C}$ one has*

$$D(\alpha f + g)(\tilde{z}) = \alpha Df(\tilde{z}) + Dg(\tilde{z}).$$

(2) *Let $f,g\colon U \to \mathbb{C}$ be holomorphic. Then the product $fg\colon U \to \mathbb{C}$ is holomorphic and for $\tilde{z} \in U$ one has (by Example 1.7 (4) and the chain rule):*

$$D(fg)(\tilde{z}) = f(\tilde{z})Dg(\tilde{z}) + g(\tilde{z})Df(\tilde{z}).$$

In particular, $\mathscr{O}(U)$ is a commutative $\mathbb{C}$-algebra.

Theorem 1.9 (Inverse Function). *Let $U \subseteq V$ open and let $f\colon U \to V'$ be holomorphic. Let $\tilde{z} \in U$ such that $Df(\tilde{z})\colon V \to V'$ is invertible. Then there exist open neighborhoods $\tilde{z} \in W \subseteq V$ and $f(\tilde{z}) \in W' \subseteq V'$ such that $f_{|W}\colon W \to W'$ is bijective and such that $g := (f_{|W})^{-1}\colon W' \to V$ is holomorphic.*

Proof. The real analogue of the inverse function theorem implies the existence of W and W' such that g is a $\mathcal{C}^1$-map with $Dg(\tilde{z}') = Df(g(\tilde{z}'))^{-1}$ for all $\tilde{z}' \in W'$. Therefore $Dg(\tilde{z}')$ is $\mathbb{C}$-linear for all $\tilde{z} \in W'$. Hence g is holomorphic. $\square$

Theorem 1.10. *Let $U \subseteq V$ be open. The following assertions are equivalent for a map $f\colon U \to V'$:*
(i) *f is holomorphic.*
(ii) *f is complex differentiable in $\tilde{z}$ for all $\tilde{z} \in U$.*
(iii) *f is analytic, i.e., locally given by a power series.*[1]
(iv) *For all $\tilde{z} \in U$, f is partially complex differentiable in $\tilde{z}$ (i.e., there exists a basis $(e_1,\dots,e_n)$ of V such that for some $\varepsilon > 0$ the maps*

$$\{\, t \in \mathbb{C}\ ;\ |t| < \varepsilon \,\} \to V', \qquad t \mapsto f(\tilde{z} + te_i)$$

are complex differentiable for all $i = 1,\dots,n$).

Analogous equivalences for real differentiable functions are completely wrong!

Proof. We will give a full proof of this theorem only for $V = V' = \mathbb{C}$ in Section 5. Here we only briefly indicate how one could proceed. Note that "(i) $\Rightarrow$ (ii) $\Rightarrow$ (iv)" are trivial. In fact, (ii) implies that (iv) holds for every basis $(e_1,\dots,e_n)$.
1st step: Show that one can assume that $V' = \mathbb{C}$. This is easy: By choosing a linear isomorphismus $V' \cong \mathbb{C}^m$ we may assume that $V' = \mathbb{C}^m$. Then $f = (f_1,\dots,f_m)$ with $f_j\colon U \to \mathbb{C}$. Then check that each assertion holds for f if and only if it holds for all f_j.
2nd step: Prove that all assertions are equivalent if $V = V' = \mathbb{C}$ (note that in this case one trivially has "(ii) $\Leftrightarrow$ (iv)", and "(iii) $\Rightarrow$ (i)" has essentially already shown in Analysis 1. Thus it suffices to show "(ii) $\Rightarrow$ (i) $\Rightarrow$ (iii)". This will be done in Section 5.
3rd step: The second step shows "(iv) $\Leftrightarrow$ (iii)". Hence it remains to show that "(iv) $\Rightarrow$ (i)". This a deep theorem due to Hartogs (see e.g.: L. Hörmander: *An introduction to complex analysis in several variables*, Theorem 2.2.8). $\square$

[1] We leave this assertion deliberately vague.

(C) Holomorphy in one variable

Notation: In this subsection we identify $\mathbb{C} \xrightarrow{\sim} \mathbb{R}^2$, $z \mapsto (x, y)$, where $x = \mathrm{Re}(z)$, $y = \mathrm{Im}(z)$.

Remark 1.11. An $\mathbb{R}$-linear map $A\colon \mathbb{C} \to \mathbb{C}$ given by a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{R})$ is $\mathbb{C}$-linear if and only if $A(iz) = iA(z)$ for all $z \in \mathbb{C}$. But the multiplication with i is given by the matrix $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ (i is the rotation by $\pi/2$). Hence A is $\mathbb{C}$-linear if and only if

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} \Leftrightarrow a = d, b = -c.$$

In this case one has:

$$A\colon \mathbb{C} \to \mathbb{C}, \qquad z \mapsto (a + ic)z. \tag{1.11.1}$$

Proposition 1.12. *Let $U \subseteq \mathbb{C}$ be open, let $f\colon U \to \mathbb{C}$ be a map, and let $\tilde{z} \in U$. Then the following assertions are equivalent.*
(i) *f is complex differentiable in $\tilde{z}$.*
(ii) *f is real differentiable (as a map $f\colon \mathbb{R}^2 \supseteq U \to \mathbb{R}^2$) and f satisfies the Cauchy-Riemann equations:*

$$\frac{\partial \mathrm{Re}(f)}{\partial x}(\tilde{z}) = \frac{\partial \mathrm{Im}(f)}{\partial y}(\tilde{z}), \qquad \frac{\partial \mathrm{Im}(f)}{\partial x}(\tilde{z}) = -\frac{\partial \mathrm{Re}(f)}{\partial y}(\tilde{z}). \tag{1.12.1}$$

(iii) *The limit*

$$f'(\tilde{z}) := \lim_{\substack{h \to 0 \\ h \in \mathbb{C}\setminus\{0\}}} \frac{f(\tilde{z} + h) - f(\tilde{z})}{h} \tag{1.12.2}$$

exists.
In this case $Df(\tilde{z})$ is the linear map $\mathbb{C} \to \mathbb{C}$, $z \mapsto f'(\tilde{z})z$ and we usually call $f'(\tilde{z})$ the (complex) derivative of f in $\tilde{z}$. *Moreover we have:*

$$f' = \frac{\partial \mathrm{Re}(f)}{\partial x} + i\frac{\partial \mathrm{Im}(f)}{\partial x} \overset{(1.12.1)}{=} \frac{\partial \mathrm{Im}(f)}{\partial y} - i\frac{\partial \mathrm{Re}(f)}{\partial y}. \tag{1.12.3}$$

Proof. "(i) $\Leftrightarrow$ (ii)": Follows from Remark 1.11 because $Df(\tilde{z})$ is given by the matrix

$$Df(\tilde{z}) = \begin{pmatrix} \frac{\partial \mathrm{Re}(f)}{\partial x} & \frac{\partial \mathrm{Re}(f)}{\partial y} \\ \frac{\partial \mathrm{Im}(f)}{\partial x} & \frac{\partial \mathrm{Im}(f)}{\partial y} \end{pmatrix}.$$

"(i) $\Leftrightarrow$ (iii)": f is complex differentiable in $\tilde{z}$
$\Leftrightarrow$ $\exists$ $\mathbb{C}$-linear map $A\colon \mathbb{C} \to \mathbb{C}$ such that

$$0 = \lim_{z \to \tilde{z}} \frac{f(z) - f(\tilde{z}) - A(z - \tilde{z})}{|z - \tilde{z}|}$$

$\Leftrightarrow \exists\, a \in \mathbb{C}$ such that

$$0 = \lim_{z \to \tilde{z}} \frac{f(z) - f(\tilde{z}) - a(z - \tilde{z})}{z - \tilde{z}} = (\lim_{z \to \tilde{z}} \frac{f(z) - f(\tilde{z})}{z - \tilde{z}}) - a.$$

Finally (1.12.3) follows from (1.11.1). □

Corollary 1.13. *$U \subseteq \mathbb{C}$ open, $f\colon U \to \mathbb{C}$. Then f is holomorphic, if* (1.12.2) *exists for all $\tilde{z} \in U$ and if $f'\colon U \to \mathbb{C}$ is continuous.*

We will see that the continuity of f' is automatic.

Example 1.14.
(1) Let $n \in \mathbb{N}_0$ and let $f\colon \mathbb{C} \to \mathbb{C}$, $f(z) = z^n$. Then

$$f'(z) = nz^{n-1}$$

(same proof as in the real case; alternatively use product rule and induction by n). Hence the linearity of the derivative shows that for $a_0, \dots, a_n \in \mathbb{C}$ and $p\colon \mathbb{C} \to \mathbb{C}$, $p(z) = a_n z^n + \dots + a_1 z + a_0$ one has

$$p'(z) = na_n z^{n-1} + \dots + a_1.$$

(2) The same proof as in real analysis shows: Let $U \subseteq \mathbb{C}$ be open, $f, g\colon U \to \mathbb{C}$ be holomorphic with $g(z) \neq 0$ for all $z \in U$. Then

$$\frac{f}{g}\colon U \to \mathbb{C}, \qquad z \mapsto \frac{f(z)}{g(z)}$$

is holomorphic, and

$$\left(\frac{f}{g}\right)'(z) = \frac{f'(z)g(z) - f(z)g'(z)}{g(z)^2}$$

for $z \in U$. In particular

$$(z^n)' = nz^{n-1}$$

for $n \in \mathbb{Z}$ and $z \in \mathbb{C}^\times$.

Proposition 1.15. *Let $U \subseteq \mathbb{C}$ be open, $f\colon U \to \mathbb{C}$ holomorphic. Then the following assertions are equivalent.*
(i) *$f'(z) = 0$ for all $z \in U$.*
(ii) *f is locally constant.*
(iii) *$\mathrm{Re}(f)\colon U \to \mathbb{R}$ is locally constant.*
(iv) *$\mathrm{Im}(f)$ is locally constant.*
(v) *$|f|\colon U \to \mathbb{R}^{\geq 0}$ is locally constant.*
(vi) *$\bar{f}\colon U \to \mathbb{C}$, $\bar{f}(z) := \overline{f(z)}$ is holomorphic.*

Recall: X topological space, M set, $t\colon X \to M$ map. Then:

$$\begin{aligned} &t \text{ locally constant} \\ :\Leftrightarrow\ &\forall x \in X\ \exists x \in U \subseteq X \text{ open, such that } t_{|U} \text{ is constant} \\ \Leftrightarrow\ &t \text{ is continuous, if we endow } M \text{ with the discrete topology} \end{aligned}$$

Proof. "(i) ⇔ (ii)": This has been proved in Analysis 2.
"(ii) ⇒ (iii) – (vi)": Obvious.
"(iii) ⇒ (i)":

$$\mathrm{Re}(f) \text{ locally constant} \Rightarrow \frac{\mathrm{Re}(f)}{\partial x} = \frac{\mathrm{Re}(f)}{\partial y} = 0 \overset{\text{CR equation}}{\Rightarrow} f' = 0.$$

"(iv) ⇒ (i)": Same argument.
"(vi) ⇒ (iii)": f, $\bar{f}$ holomorphic ⇒ $u := \mathrm{Re}(f) = (f+\bar{f})/2$ holomorphic. But $\mathrm{Im}(u) = 0$ and thus we can apply "(iv) ⇒ (ii)" for u instead of f to see that u is locally constant.
"(v) ⇒ (vi)": Clear if $f = 0$. As $|f|$ is locally constant we can assume that $f(z) \neq 0$ for all $z \in U$. Then: $1/f$ is a holomorphic function ⇒ $\bar{f} = |f|^2/f$ holomorphic. □

2 Path integrals

(A) Vector valued 1-forms

Notation: In this section we denote by V and W finite-dimensional $\mathbb{R}$-vector spaces. Later we will be mainly interested in the case $V = W = \mathbb{C}$.

Definition 2.1. Let $U \subseteq V$ be a subset. A *W-valued 1-form on U* is a map

$$\omega \colon U \to \mathrm{Hom}_{\mathbb{R}}(V, W).$$

Note that it makes sense to say that ω is continuous. Moreover if $U \subseteq V$ is open, then it makes sense to say that ω is a $\mathcal{C}^k$-map ($k \in \mathbb{N}_0 \cup \{\infty\}$).

In the language of "Reelle Analysis" it would have been better to define that ω is a map that sends $p \in U$ to an alternating 1-multilinear form $\omega(p) \colon T_p(U) \to W$ and to remark that $T_p(U) = V$ for all $p \in U$.

Example 2.2. Let $U \subseteq V$ be open and let $F \colon U \to W$ be real differentiable. Then $dF := DF$ is a W-valued 1-form ($DF(p) \in \mathrm{Hom}_{\mathbb{R}}(V, W)$ for all $p \in U$).

Remark 2.3 (Coordinates). We now assume $V = \mathbb{R}^m$. For $i = 1, \dots, m$ we call

$$x^i \colon \mathbb{R}^m \to \mathbb{R}, \qquad (p_1, \dots, p_m) \mapsto p_i$$

the *coordinate functions*.
Then x^i is $\mathbb{R}$-linear and $dx^i(p) = x^i \in \mathrm{Hom}_{\mathbb{R}}(\mathbb{R}^m, \mathbb{R})$ for all $p \in \mathbb{R}^m$. Hence for all $p \in \mathbb{R}^m$, $(dx^1(p), \dots, dx^m(p))$ is a basis of $\mathrm{Hom}_{\mathbb{R}}(\mathbb{R}^m, \mathbb{R})$ and for every $\mathbb{R}$-valued 1-form ω on $U \subseteq \mathbb{R}^m$ we have

$$\omega(p) = f_1(p)dx^1(p) + \cdots + f_m(p)dx^m(p)$$

for unique functions $f_i \colon U \to \mathbb{R}$.
More generally, every $\alpha \in \mathrm{Hom}_{\mathbb{R}}(\mathbb{R}^m, W)$ is of the form $\alpha = w_1 dx^1(p) + \cdots + w_m dx^m(p)$ for unique $w_1, \dots, w_m \in W$, where

$$w_i dx^i(p) \colon \mathbb{R}^m \to W, \qquad y = (y_1, \dots, y_m) \mapsto dx^i(p)(y) w_i = y_i w_i.$$

Hence every W-valued 1-form ω on $U \subseteq \mathbb{R}^m$ is of the form

$$\omega = f_1 dx^1 + \cdots + f_m dx^m$$

for unique functions $f_i \colon U \to W$ und we have

$$(f_i dx^i)(p)(y_1, \ldots, y_m) = y_i f_i(p). \tag{2.3.1}$$

Moreover ($k \in \mathbb{N}_0 \cup \{\infty\}$):

$$\omega \text{ is a } \mathcal{C}^k \text{ 1-form} \Leftrightarrow f_1, \ldots, f_m \text{ are } \mathcal{C}^k \text{ maps.}$$

Remark 2.4. Let $V = \mathbb{R}^m$, $U \subseteq \mathbb{R}^m$ open, $F \colon U \to W$ real differentiable. Then one has

$$dF = \frac{\partial F}{\partial x_1} dx^1 + \cdots + \frac{\partial F}{\partial x_m} dx^m,$$

where $\frac{\partial F}{\partial x_i} \colon U \to W$ denotes the i-th partial derivative of F.

Definition 2.5. Let $U \subseteq \mathbb{R}^m$ be open and let $\omega = \sum_{i=1}^m f_j dx^j \colon U \to \mathrm{Hom}_{\mathbb{R}}(V, W)$ be a W-valued 1-form.

(1) ω is called *exact* if there exists a real differentiable function $F \colon U \to W$ such that $\omega = dF$.

(2) ω is called *closed* if ω is continuously differentiable and for all $i, j \in \{1, \ldots, m\}$ one has

$$\frac{\partial f_i}{\partial x_j} = \frac{\partial f_j}{\partial x_i}.$$

Proposition 2.6. *Let $U \subseteq \mathbb{R}^m$ offen and let ω be a continuously differentiable W-valued 1-form on U. Then:*

$$\omega \text{ exact} \Rightarrow \omega \text{ closed.}$$

Proof. This has been proved in Analysis 2: It is a direct calculation: As ω is exact, $\omega = dF$ for a $\mathcal{C}^2$-map $F \colon U \to W$. And we know from Analysis 2 that

$$\frac{\partial^2 F}{\partial x_i \partial x_j} = \frac{\partial^2 F}{\partial x_j \partial x_i}.$$

□

(B) Path integrals

Notation: Let $a, b \in \mathbb{R}$, $a < b$. Let V be a finite-dimensional $\mathbb{R}$-vector space, $W = \mathbb{R}^n$, $n \in \mathbb{N}$.

Definition 2.7.

(1) Let X be a topological space. A continuous map $\gamma \colon [a, b] \to X$ is called *path in X* (German: *Weg in X*). The point $\gamma(a)$ is called the *startpoint*, $\gamma(b)$ is called the *endpoint* of γ. We say that γ is a *path from $\gamma(a)$ to $\gamma(b)$*. We also set:

$$\{\gamma\} := \gamma([a, b]) \subseteq X.$$

(2) The path γ is called *closed* or a *loop* (German: *Schleife*), if $\gamma(a) = \gamma(b)$.
(3) A path $\gamma\colon [a,b] \to X$ is called *constant*, if there exists $x_0 \in X$ such that $\gamma(t) = x_0$ for all $t \in [a,b]$. We then denote γ by ε_{x_0}.
(4) Let $X = V$. A path $\gamma\colon [a,b] \to V$ is called $\mathcal{C}^k$, if
   (a) $\gamma_{|(a,b)}$ is a $\mathcal{C}^k$-map. For $l \leq k$ we denote by $\gamma^{(l)}\colon (a,b) \to V$ its l-th derivative.
   (b) For all $l \in \mathbb{N}$ with $l \leq k$ the limits

$$\gamma^{(l)}(a) := \lim_{t \searrow a} \gamma^{(l)}(t), \qquad \text{and} \qquad \gamma^{(l)}(b) := \lim_{t \nearrow b} \gamma^{(l)}(t)$$

   exist.
(5) A path $\gamma\colon [a,b] \to V$ is called *piecewise* $\mathcal{C}^k$ if there exist $a = t_0 < t_1 < \cdots < t_r = b$ such that $\gamma_{|[t_{i-1},t_i]}$ is a $\mathcal{C}^k$-path for all $i = 1, \ldots, r$.

Definition 2.8. Let $\gamma\colon [a,b] \to V$ be a $\mathcal{C}^1$-path. Let $\omega\colon \{\gamma\} \to \operatorname{Hom}(V,W)$ be a continuous 1-form. Define

$$\int_\gamma \omega := \int_a^b \omega(\gamma(t))(\gamma'(t))\,dt \qquad \left(= \int_{[a,b]} \gamma^*\omega \right)^2 \qquad \in W.$$

Here we calculate the integral for every component of W.
More generally, if γ is piecewise $\mathcal{C}^1$ with $a = t_0 < t_1 < \cdots < t_r = b$ such that $\gamma_{|[t_{i-1},t_i]}$ is a $\mathcal{C}^1$-path, then set

$$\int_\gamma \omega := \sum_{i=1}^r \int_{\gamma_{|[t_{i-1},t_i]}} \omega.$$

If $S \subseteq V$ with $\{\gamma\} \subseteq S$ and $\omega\colon S \to \operatorname{Hom}(V,W)$ continuous, we write $\int_\gamma \omega$ instead of $\int_\gamma \omega_{|\{\gamma\}}$.

Example 2.9. Let $x \in V$ and let $\varepsilon_x\colon [a,b] \to U$ be the constant path with value x. Then $\varepsilon_x'(t) = 0$ for all $t \in [a,b]$ and hence

$$\int_{\varepsilon_x} \omega = 0$$

for all ω.

Remark 2.10. Let $\gamma\colon [a,b] \to U$ be a piecewise $\mathcal{C}^1$-path and let $\omega, \eta\colon \{\gamma\} \to \operatorname{Hom}(V,W)$ be a continuous 1-forms.
(1) For all $\lambda \in \mathbb{R}$ one has

$$(*) \qquad \int_\gamma (\lambda\omega + \eta) = \lambda \int_\gamma \omega + \int_\gamma \eta.$$

   If W is a $\mathbb{C}$-vector space, (*) holds also for $\lambda \in \mathbb{C}$.

[2] In the language of "Reelle Analysis"

(2) Let $c, d \in \mathbb{R}$ with $c < d$ and let $\varphi\colon [c,d] \to [a,b]$ be $\mathcal{C}^1$ and bijective with $\pm\varphi'(t) \geq 0$ for all $t \in [c,d]$. Then
$$\int\limits_{\gamma\circ\varphi} \omega = \pm \int\limits_{\gamma} \omega.$$

Proof. This has been proved in Analysis 2. □

Example 2.11. Let $\gamma\colon [a,b] \to V = \mathbb{R}^m$, $t \mapsto (\gamma_1(t), \dots, \gamma_m(t))$ be a $\mathcal{C}^1$-path. Let $\omega = \sum_{i=1}^m f_i dx^i$ a W-valued 1-form with $f_i\colon \{\gamma\} \to W$ continuous. Then
$$\begin{aligned}\int\limits_{\gamma} \omega &= \int\limits_a^b \omega(\gamma(t))(\gamma'(t))\,dt \\ &= \int\limits_a^b \sum_{i=1}^m (f_i dx^i)(\gamma(t))(\gamma'(t))\,dt \\ &\overset{(2.3.1)}{=} \int\limits_a^b \sum_{i=1}^m \gamma_i'(t) f_i(\gamma(t))\,dt.\end{aligned}$$

Example 2.12. Let $W = \mathbb{C}$, $V = \mathbb{R}^2$ with coordinates x and y. Consider the 1-form on $\mathbb{R}^2 \setminus \{0\}$
$$\omega = \frac{1}{x+iy}(dx + idy) = \frac{(x-iy)dx + (y+ix)dy}{x^2+y^2}.$$
Let $\gamma\colon [0, 2\pi] \to \mathbb{R}^2$, $t \mapsto (\cos t, \sin t)$. Then one has
$$\begin{aligned}\int\limits_{\gamma} \omega &= \int\limits_0^{2\pi} (-\sin(t)(\cos(t) - i\sin(t)) + \cos(t)(\sin(t) + i\cos(t)))\,dt \\ &= \int\limits_0^{2\pi} i(\sin^2(t) + \cos^2(t))\,dt \\ &= 2\pi i.\end{aligned}$$

Remark 2.13. Let $\gamma\colon [a,b] \to V$ be a path. Let $\varphi\colon [0,1] \to [a,b]$, $\varphi(t) = a + (b-a)t$. Then:
$$\gamma \text{ (piecewise) } \mathcal{C}^k \Leftrightarrow \gamma \circ \varphi\colon [0,1] \to V \text{ (piecewise) } \mathcal{C}^k.$$
Moreover $\int_\gamma \omega = \int_{\gamma\circ\varphi} \omega$ (if γ is piecewise $\mathcal{C}^1$, ω continuous W-valued 1-form).

Upshot: Can usually assume that paths are defined on $[0,1]$.

Definition 2.14. Let X be a topological space.
(1) Let $\gamma\colon [0,1] \to X$ be a path. Define
$$\gamma^-\colon [0,1] \to X, \gamma^-(t) = \gamma(1-t)$$
the *inverse path*.

(2) Let $\gamma, \delta\colon [0,1] \to X$ be two paths with $\gamma(1) = \delta(0)$. Define a path

$$\gamma \cdot \delta\colon [0,1] \to X, \qquad t \mapsto \begin{cases} \gamma(2t), & 0 \le t \le 1/2; \\ \delta(2t-1), & 1/2 \le t \le 1. \end{cases}$$

Remark 2.15. Let $\gamma, \delta\colon [0,1] \to V$ piecewise $\mathcal{C}^1$-paths. Then Remark 2.10 shows

$$\int\limits_{\gamma^-} \omega = -\int\limits_{\gamma} \omega, \qquad \text{for all } \omega\colon \{\gamma\} \to \operatorname{Hom}(V,W) \text{ continuous,}$$

$$\int\limits_{\gamma\cdot\delta} \omega = \int\limits_{\gamma} \omega + \int\limits_{\delta} \omega, \qquad \text{for all } \omega\colon \{\gamma\} \cup \{\delta\} \to \operatorname{Hom}(V,W) \text{ continuous,}$$

where for the second equality we also assume that $\gamma(1) = \delta(0)$.

Proposition 2.16. *Let $U \subseteq V$ be open, $F\colon U \to W$ a $\mathcal{C}^1$-map, and let $\gamma\colon [a,b] \to U$ be a piecewise $\mathcal{C}^1$-path. Then*

$$\int\limits_{\gamma} dF = F(\gamma(b)) - F(\gamma(a)).$$

In particular $\int_\gamma \omega = 0$ if ω is a continuous exact 1-form on U and γ is closed.

Proof. This has been proved in Analysis 2. $\square$

Example 2.17. Let $V = \mathbb{R}^2$ with coordinates x and y, and let $W = \mathbb{C}$.
(1) (Stupid example) The 1-form $\omega = (x + iy^2)dx + x^3dy$ on $\mathbb{R}^2$ is not closed: Set

$$f_x\colon \mathbb{R}^2 \to \mathbb{C}, \qquad f_x(x,y) = x + iy^2;$$
$$f_y\colon \mathbb{R}^2 \to \mathbb{C}, \qquad f_y(x,y) = x^3.$$

Then

$$\frac{\partial f_x}{\partial y}(x,y) = 2iy, \qquad \frac{\partial f_y}{\partial x}(x,y) = 3x^2.$$

(2) (Intelligent example) The 1-form on $\mathbb{R}^2 \setminus \{0\}$ (already considered in Example 2.12)

$$\omega = \frac{1}{x+iy}(dx + idy)$$

is closed (easy calculation or see 4.2), but we have seen in Example 2.12 that $\int_\gamma \omega \neq 0$ for the closed path $\gamma\colon [0,2\pi] \to \mathbb{C}$, $t \mapsto (\cos t, \sin t)$. This shows that ω is not exact (Proposition 2.16).

(C) Limits and Integral

Definition 2.18. Let X be a topological space, (Y, d) metric space. Let $(f_n)_n$ be a sequence of functions $f_n\colon X \to Y$, and let $f\colon X \to Y$. One says that

(1) $(f_n)_n$ *converges locally uniformly to* f, if for all $x \in X$ there exists $x \in U \subseteq X$ open such that

$$\sup_{x \in U} d(f_n(x), f(x)) \overset{n\to\infty}{\longrightarrow} 0,$$

i.e., $(f_{n|U})_n$ converges uniformly to $f_{|U}$.

(2) $(f_n)_n$ *converges compactly to* f, if for every compact subspace K of X one has

$$\sup_{x \in K} d(f_n(x), f(x)) \overset{n\to\infty}{\longrightarrow} 0,$$

i.e., $(f_{n|K})_n$ converges uniformly to $f_{|K}$.

Remark 2.19. Notation as in Definition 2.18.

(1) If $(f_n)_n$ converges locally uniformly to f, then $(f_n)_n$ converges compactly to f (every compact set K can be covered by finitely many U's as above).

(2) Now assume that $X \subseteq \mathbb{R}^d$ open ($d \in \mathbb{N}$). If $(f_n)_n$ converges compactly to f, then $(f_n)_n$ converges locally uniformly to f ($\forall\, x \in X\ \exists\, x \in U \subseteq K \subseteq X$ with $U \subseteq X$ open and K compact).

Assertion (2) holds more generally, if X is locally compact.
Upshot: For $X \subseteq \mathbb{R}^d$ open (or, more generally, for X locally compact) locally uniform convergence and compact convergence are equivalent.

Proposition 2.20. *Let $S \subseteq V$ be a subspace, let $\gamma\colon [a, b] \to V$ be a piecewiese $\mathcal{C}^1$-path with $\{\gamma\} \subseteq S$. Let $(\omega_n)_{n\in\mathbb{N}}$ be a sequence of continuous 1-forms $\omega_n\colon S \to \mathrm{Hom}_{\mathbb{R}}(V, W)$ which converge locally uniformly to a 1-form $\omega\colon S \to \mathrm{Hom}_{\mathbb{R}}(V, W)$. Then ω is continuous, and*

$$\lim_{n\to\infty} \int_\gamma \omega_n = \int_\gamma \omega.$$

Proof. By Analysis 2 we already know that ω is continuous. As $(\omega_n)_n$ converges locally uniformly and γ' is bounded, $(t \mapsto \omega_n(\gamma(t))\gamma'(t))_n$ converges locally uniformly and hence uniformly because $[a, b]$ is compact. Hence the claim follows that the integral commutes with uniform limit of functions (Analysis 1). □

(D) Digression: Connected and path-connected spaces

Definition 2.21. Let $X \neq \emptyset$ be a topological space.

(1) X is called *connected* if for every open and closed $\emptyset \neq Z \subseteq X$ one has $X = Z$.

(2) X is called *path-connected* if for all $x, y \in X$ there exists a continuous path $\gamma\colon [0, 1] \to X$ with $\gamma(0) = x$ and $\gamma(1) = y$.

Warning: According to this definition the empty space is not connected.

Proposition 2.22.

(1) *Every path-connected toplogical space is connected.*

(2) *Let V be finite-dimensional $\mathbb{R}$-vector space, $U \subseteq V$ be open and connected. Then for any $x, y \in U$ there exists a piecewise $\mathcal{C}^\infty$-path $\gamma\colon [0,1] \to U$ with $\gamma(0) = x$ and $\gamma(1) = y$. (In particular: U is path-connected.)*

Hence we see that if $U \subseteq V$ open, V finite-dimensional $\mathbb{R}$-vector space, then:

$$U \text{ connected} \Leftrightarrow U \text{ path-connected.}$$

In general there exist connected topological spaces (even subspaces of $\mathbb{R}^2$) which are not path-connected.

Proof. This has been proved in Analysis 2. □

Definition 2.23. An open (path-)connected subspace of a finite-dimensional $\mathbb{R}$-vector space is called *domain* (German: *Gebiet*).

Proposition 2.24. *Let $\emptyset \neq X$ be a topological space. Then X is connected if and only if for every set M every locally constant map $t\colon X \to M$ is constant.*

Proof. Let X be connected. Choose $x \in X$. If t is locally constant, then $t^{-1}(t(x))$ is open and closed in X and hence $= X$. Therefore $t(y) = t(x)$ for all $y \in X$.
Conversely, assume that X is not connected. Then there exists $\emptyset \neq Y \subsetneq X$ open and closed. Hence the characteristic function of Y

$$\chi_Y\colon X \to \{0,1\}, \qquad x \mapsto \begin{cases} 1, & x \in Y; \\ 0, & x \notin Y \end{cases}$$

is locally constant but not constant. □

Remark 2.25. Let $f\colon X \to Y$ be a continuous surjective map of topological spaces. If X is (path-)connected, then Y is (path-)connected.

Proof. "X connected $\Rightarrow$ Y connected": $\emptyset \neq Z \subseteq Y$ open and closed $\Rightarrow$ $f^{-1}(Z)$ open and closed (because f is continuous) and $f^{-1}(Z) \neq \emptyset$ (because f is surjective). As X is connected, $f^{-1}(Z) = X$. Hence

$$Z \overset{f \text{ surjective}}{=} f(f^{-1}(Z)) = f(X) = Y.$$

"X path-connected $\Rightarrow$ Y path-connected": Let $y, y' \in Y$.
f surjective $\Rightarrow \exists\, x, x' \in X$ with $f(x) = y$ and $f(x') = y'$. Hence

$$\begin{aligned} X \text{ path-connected} &\Rightarrow \exists\, \gamma\colon [0,1] \to X \text{ continuous with } \gamma(0) = x,\ \gamma(1) = x' \\ &\Rightarrow f \circ \gamma\colon [0,1] \to Y \text{ continuous with } f(\gamma(0)) = y,\ f(\gamma(1)) = y'. \end{aligned}$$ □

Remark and Definition 2.26. Let X be a topological space. Define on X the relation

$$x \sim y \Leftrightarrow \exists\, \gamma\colon [0,1] \to X \text{ path such that } \gamma(0) = x,\ \gamma(1) = y.$$

This is an equivalence relation and the equivalence classes are called the *path components of X* (i.e., $x, y \in X$ are in the same path-component if and only if x and y can be connected by a path). The set of path components of X is denoted by $\pi_0(X)$.
Every path-component is path-connected.

Proposition 2.27. *Let V be finite-dimensional $\mathbb{R}$-vector space, let $X \subseteq V$ be open, and let $Z \subseteq X$ be a path-component. Then Z is open and closed in X.*

For an arbitary topological space X a path component of X is in general neither open nor closed.

Proof. As X is open in a finite-dimensional $\mathbb{R}$-vector space, every point $z \in Z$ has an open path-connected neighborhood U (e.g., some small open ball) in X. Hence $U \subseteq Z$. This shows that Z is open. Hence

$$X \setminus Z = \bigcup_{Z' \neq Z \text{ path-comp. of } X} Z'$$

is also open. $\square$

The same proof is valid for every topological space X such that for all $x \in X$ there exists an open path-connected neighborhood of X.

(E) Existence of primitives

The following result gives criteria for a 1-form ω to be exact, i.e. to answer the question, when a function F exists with $dF = \omega$ (a primitive (German: Stammfunktion) of ω).

Theorem 2.28. *Let V be finite-dimensional $\mathbb{R}$-vector space, $W = \mathbb{R}^n$. Let $U \subseteq V$ be open, $\omega\colon U \to \mathrm{Hom}(V, W)$ a continuous 1-form. Then the following assertions are equivalent:*
(i) *ω is exact.*
(ii) *For all $x, y \in U$ we have: Given piecewise $\mathcal{C}^1$-paths $\gamma_i\colon [a_i, b_i] \to U$, $i = 1, 2$ with $\gamma_1(a_1) = \gamma_2(a_2) = x$ and $\gamma_1(b_1) = \gamma_2(b_2) = y$. Then*

$$\int_{\gamma_1} \omega = \int_{\gamma_2} \omega.$$

(iii) *For every closed piecewise $\mathcal{C}^1$-path $\gamma\colon [a, b] \to U$ one has*

$$\int_{\gamma} \omega = 0.$$

Proof. Replacing U by its path-components, we may assume that U is a domain. Then the result has been proved in Analysis 2 for $W = \mathbb{R}$. The proof in the general case is the same.
Recall that for the essential step "(ii) $\Rightarrow$ (i)" one obtains $F\colon U \to W$ as follows. Fix $x_0 \in U$ and define

$$F\colon U \to W, \qquad F(x) := \int_{x_0}^{x} \omega := \int_{\gamma} \omega,$$

where γ is any piecewise $\mathcal{C}^1$-path with startpoint x_0 and endpoint x. This is well defined by (ii). One has $dF = \omega$ (!).
Note that "(i) $\Rightarrow$ (ii)" is a direct consequence of Proposition 2.16. $\square$

3 Homotopy

(A) Homotopy and simply connected spaces

Notation: Let X be a topological space, $a, b \in \mathbb{R}$, $a < b$.

Definition 3.1. Let $\gamma_0, \gamma_1 \colon [a,b] \to X$ paths.

(1) Assume that $\gamma_0(a) = \gamma_1(a)$ and $\gamma_0(b) = \gamma_1(b)$. A *homotopy of* γ_0 *and* γ_1 *in* X is a continuous map $H \colon [a,b] \times [0,1] \to X$ such that
   (a) $H(t,0) = \gamma_0(t)$ for all $t \in [a,b]$,
   $H(t,1) = \gamma_1(t)$ for all $t \in [a,b]$.
   (b) $H(a,s) = \gamma_0(a) = \gamma_1(a)$, $H(b,s) = \gamma_0(b) = \gamma_1(b)$ for alle $s \in [0,1]$.
   Thus for all $s \in [0,1]$ the paths $\gamma_s \colon [a,b] \to X$, $t \mapsto H(t,s)$ have all the same endpoints as γ_0 and γ_1.
   If there exists a homotopy H of γ_0 and γ_1 in X, we call γ_0 and γ_1 *homotopic*. We then write $\gamma \simeq \delta$ or $H \colon \gamma \simeq \delta$.

(2) Assume that γ_0 and γ_1 are loops (but not necessarily with same endpoints). A *loop homotopy of* γ_0 *and* γ_1 *in* X is a continuous map $H \colon [a,b] \times [0,1] \to X$ such that
   (a) $H(t,0) = \gamma_0(t)$ for all $t \in [a,b]$,
   $H(t,1) = \gamma_1(t)$ for all $t \in [a,b]$.
   (b) $H(a,s) = H(b,s)$ for alle $s \in [0,1]$.
   Thus for all $s \in [0,1]$ the paths $\gamma_s \colon [a,b] \to X$, $t \mapsto H(t,s)$ are loops (but the endpoints may change).
   If there exists a loop homotopy H of γ_0 and γ_1 in X, we call γ_0 and γ_1 *loop homotopic*.

(3) A path $\gamma \colon [a,b] \to X$ is called *null-homotopic in* X if it is homotopic in X to a constant path ($\Rightarrow \gamma$ is a loop).

Example 3.2. Let $\gamma \colon [0, 2\pi] \to \mathbb{C}$, $\gamma(t) = e^{it}$. Then γ is a $\mathcal{C}^\infty$-loop in $\mathbb{C}$. We will see that γ is null-homotopic in $\mathbb{C}$ (Remark 3.9 will show that all loops in $\mathbb{C}$ are null-homotopic in $\mathbb{C}$) but that γ is not null-homotopic in $\mathbb{C}^\times$ (this will follow from Theorem 3.12 and Example 2.17 (2)). Note that this is graphically clear.

Remark 3.3. Let $\mathcal{C}([a,b], X)$ be the set of all paths $[a,b] \to X$. Then the relation "homotopic" $\simeq$ is an equivalence relation on $\mathcal{C}([a,b], X)$.
The relation "loop homotopic" on the sets of all loops in X is an equivalence relation.

Proof. Let us show that $\simeq$ is an equivalence relation (the proof for "loop homotopic" is the same). Reflexivity: Clear
Symmetry: H homotopy of γ and δ. Then

$$H^- \colon [a,b] \times [0,1], \qquad H^-(t,s) := H(t, 1-s)$$

is a homotopy of δ and γ.
Transitivity: Let $H' \colon \gamma \simeq \delta$, $H'' \colon \delta \simeq \varepsilon$. Then

$$H \colon [a,b] \times [0,1], \qquad H(t,s) := \begin{cases} H'(t, 2s), & 0 \le s \le 1/2; \\ H''(t, 2s-1), & 1/2 \le s \le 1. \end{cases}$$

is a homotopy of γ and ε. $\square$

Remark 3.4. Let $\gamma, \gamma_1, \gamma_2, \gamma_3, \delta_1, \delta_2 \colon [0,1] \to X$ be paths in X.
(1) Let $\varphi \colon [0,1] \to [0,1]$ be continuous with $\varphi(0) = 0$ and $\varphi(1) = 1$. Then $\gamma \simeq \gamma \circ \varphi$.
(2) Set $x_0 := \gamma(0)$, $x_1 := \gamma(1)$. Then

$$\varepsilon_{x_0} \cdot \gamma \simeq \gamma \simeq \gamma \cdot \varepsilon_{x_1}.$$

(3) $\gamma_1 \simeq \gamma_2 \Rightarrow \gamma_1^- \simeq \gamma_2^-$.
(4) Assume $\gamma_i(1) = \delta_i(0)$ for $i = 1, 2$. Then $\gamma_1 \simeq \gamma_2$, $\delta_1 \simeq \delta_2 \Rightarrow \gamma_1 \cdot \delta_1 \simeq \gamma_2 \cdot \delta_2$.
(5) $\gamma_1 \simeq \gamma_2 \Leftrightarrow \gamma_1 \cdot \gamma_2^-$ null-homotopic.
(6) Assume $\gamma_1(1) = \gamma_2(0)$ and $\gamma_2(1) = \gamma_3(0)$. Then

$$\gamma_1 \cdot (\gamma_2 \cdot \gamma_3) \simeq (\gamma_1 \cdot \gamma_2) \cdot \gamma_3.$$

(7) Let γ and δ be loops, $x := \gamma(0)$, $y := \delta(0)$. Then γ and δ are loop homotopic if and only if there exists a path $\sigma \colon [0,1] \to X$ with $\sigma(0) = x$ and $\sigma(1) = y$ such that γ and $\sigma \cdot \delta \cdot \sigma^-$ are homotopic.

Proof. Exercise
For (7): Let H be a loop homotopy of γ and δ and define a path $\sigma \colon [0,1] \to X$, $\sigma(s) := H(0,s) = H(1,s)$. Then $\sigma(0) = x$ and $\sigma(1) = y$. Define a homotopy of γ and $\sigma \cdot \delta \cdot \sigma^-$ by

$$H' \colon [0,1] \times [0,1], \qquad H'(t,s) := \begin{cases} \sigma(3ts), & 0 \le t \le 1/3; \\ H(3t-1, s), & 1/3 \le t \le 2/3; \\ \sigma((3-3t)s), & 2/3 \le t \le 1. \end{cases}$$

Conversely, assume that there exists a path σ with $\sigma(0) = x$ and $\sigma(1) = y$ such that there exists a homotopy $H' \colon \gamma \simeq \sigma \cdot \delta \cdot \sigma^-$. Then γ and $\sigma \cdot \delta \cdot \sigma^-$ are also loop homotopic. As the relation of being loop homotopic is transitive, it suffices to show that δ and $\sigma \cdot \delta \cdot \sigma^-$ are loop homotopic. Define a loop homotopy of δ and $\sigma \cdot \delta \cdot \sigma^-$ by

$$H \colon [0,1] \times [0,1], \qquad H(t,s) := \begin{cases} \sigma(1 - s + 3ts), & 0 \le t \le 1/3; \\ \delta(3t-1), & 1/3 \le t \le 2/3; \\ \sigma(1 + (2-3t)s), & 2/3 \le t \le 1. \end{cases}$$

$\square$

Example 3.5. Let $X = \mathbb{C}$, $\tilde{z} \in \mathbb{C}$, let $r \in \mathbb{R}^{>0}$. Recall that we defined

$$B_r(\tilde{z}) := \{ z \in \mathbb{C} \; ; \; |z - \tilde{z}| < r \}.$$

Let

$$\gamma_r^{\tilde{z}} \colon [0,1] \to \mathbb{C}, \qquad t \mapsto \tilde{z} + r \exp(2\pi i t).$$

This is a loop. Instead of $\gamma_r^{\tilde{z}}$ we also write $\partial B_r(\tilde{z})$.
Let $z_0 \in B_r(\tilde{z})$, $\varepsilon > 0$. Then $\partial B_r(\tilde{z})$ and $\partial B_\varepsilon(z_0)$ are loop homotopic in $\mathbb{C} \setminus \{z_0\}$.

Proof. As "loop homotopic" is an equivalence relation, we may assume that $\varepsilon > 0$ is small enough such that $B_\varepsilon(z_0) \subseteq B_r(\tilde{z})$.
For $z \in \partial B_r(\tilde{z})$ let $z_\varepsilon \in \partial B_\varepsilon(z_0)$ be the point of intersection of a line through z and z_0 with $\partial B_\varepsilon(z_0)$, i.e.,

$$z_\varepsilon := z_0 + \varepsilon \frac{z - z_0}{|z - z_0|}.$$

Define a loop homotopy of $\partial B_r(\tilde{z})$ and $\partial B_\varepsilon(z_0)$ by

$$H\colon [0,1] \times [0,1] \to \mathbb{C}, \qquad (t,s) \mapsto s\gamma_r^{\tilde{z}}(t)_\varepsilon + (1-s)\gamma_r^{\tilde{z}}(t).$$

Then $B_\varepsilon(z_0) \cap H([0,1] \times [0,1]) = \emptyset$. In particular, H is a loop homotopy in $\mathbb{C} \setminus \{z_0\}$. □

Definition and Remark 3.6. A topological space X is called *simply connected* if it is path-connected and if one the following equivalent conditions is satisfied.
(i) Any two paths $\gamma, \delta\colon [a,b] \to X$ with the same startpoint and the same endpoint are homotopic to each other.
(ii) Every loop in X is null-homotopic.
(iii) Any two loops in X are loop homotopic.

Proof. "(i) $\Leftrightarrow$ (ii)": Remark 3.4 (5).
"(ii) $\Leftrightarrow$ (iii)": Remark 3.4 (7). □

Remark 3.7. Let X and Y be topological spaces, $f\colon X \to Y$ a homeomorphism. Then X is simply connected if and only if Y is simply connected.

Proof. Symmetry in X and Y $\Rightarrow$ It suffices to show: "X simply connected $\Rightarrow$ Y simply connected": Let $\delta\colon [a,b] \to Y$ be a closed path. Then $\gamma := f^{-1} \circ \delta\colon [a,b] \to X$ is a closed path and hence there exists a homotopy to the constant path $H\colon \gamma \simeq \varepsilon_{x_0}$, where $x_0 := \gamma(0) = \gamma(1)$. Then $f \circ H\colon \delta \simeq \varepsilon_{y_0}$, where $y_0 := \delta(0) = \delta(1)$. □

Definition 3.8. Let V be an $\mathbb{R}$-vector space. A subset $S \subset V$ is called *star-shaped* (German: *sternförmig*), if there exists a point $x_0 \in V$ such that for all $x \in V$ one has

$$\{\, x_0 + t(x - x_0) \;;\; 0 \le t \le 1 \,\} \subseteq S.$$

(The left hand side is the line segment from x_0 to x.) Then x_0 is called a *star center* (German: *Sternzentrum*).

Remark 3.9. Let V be a finite-dimensional $\mathbb{R}$-vector space and let $\emptyset \neq X \subseteq V$ be a subspace. Then

$$X \text{ convex} \Rightarrow X \text{ star-shaped} \Rightarrow X \text{ simply connected} \Rightarrow X \text{ path-connected}.$$

Proof. The first implication is clear: In a convex set every point is a star center. The last implication is by definition.
Let X be star-shaped with star center x. Let $\gamma\colon [a,b] \to X$ be a loop. Then

$$H\colon [a,b] \times [0,1] \to X, \qquad H(t,s) := x + s(\gamma(t) - x)$$

is loop homotopy of γ and ε_x in X. □

Example 3.10.

(1) Let V be finite-dimensional $\mathbb{R}$-vector space, $\|\cdot\|$ be a norm on V, $v_0 \in V$, $r \in \overline{\mathbb{R}}^{>0}$. Then $B_r(v_0) = \{ v \in V \ ; \ \|v - v_0\| < r \}$ is convex and hence simply connected. In particular: V is simply connected.

(2) $\mathbb{C} \setminus \{ x \in \mathbb{R} \ ; \ x \leq 0 \}$ is star-shaped with star center 1 and hence simply connected.

(B) Homotopy invariance

Notation: In this subsection let $W = \mathbb{R}^n$, $U \subseteq V := \mathbb{R}^m$ be open and $\omega \colon U \to \mathrm{Hom}_{\mathbb{R}}(V, W)$ a continuously differentiable W-valued 1-form on U.

Lemma 3.11 (Poincaré lemma, local version). *Let $U \subseteq \mathbb{R}^m$ be star-shaped, let $\omega \colon U \to \mathrm{Hom}_{\mathbb{R}}(V, W)$ be closed. Then ω is exact.*

Proof. After a possible translation, we may assume that the star center is $0 \in \mathbb{R}^m$ (to simplify the notation). Let $\omega = \sum_{i=1}^m f_i dx^i$ with $f_i \colon U \to W$ a $\mathcal{C}^1$-map. Define

$$F \colon U \to W, \qquad F(x) := \int_0^1 \left(\sum_{i=1}^m f_i(tx) x_i \right) dt, \qquad x = (x_1, \dots, x_m) \in U.$$

This is well-defined because $\{ tx \ ; \ 0 \leq t \leq 1 \} \subseteq U$ for all $x \in U$.
We claim that $dF = \omega$. We have to show that $\frac{\partial F}{\partial x_j} = f_j$ for $j = 1, \dots, m$. As the map $(t, x) \mapsto \sum_{i=1}^m f_i(tx) x_i$ is $\mathcal{C}^1$, we may interchange integral and derivative. Hence:

$$\begin{aligned}
\frac{\partial F}{\partial x_j}(x) &= \int_0^1 \left(\sum_{i=1}^m \frac{\partial}{\partial x_j} (f_i(tx) x_i) \right) dt \\
&= \int_0^1 \Big(f_j(tx) + t \Big(\sum_{i=1}^m x_i \frac{\partial f_i}{\partial x_j}(tx) \Big)\Big) \, dt \\
&\overset{\omega \text{ closed}}{=} \int_0^1 f_j(tx) \, dt + \int_0^1 t \left(\sum_{i=1}^m x_i \frac{\partial f_j}{\partial x_i}(tx) \right) dt \\
&\overset{(*)}{=} \int_0^1 f_j(tx) \, dt + t f_j(tx) \Big|_0^1 - \int_0^1 f_j(tx) \, dt \\
&= f_j(x),
\end{aligned}$$

where (*) holds by partial integration because

$$\frac{\partial}{\partial t}(f_j(tx)) \overset{\text{chain rule}}{=} \left(\frac{\partial f_j}{\partial x_1}(tx), \dots, \frac{\partial f_j}{\partial x_m}(tx) \right) \begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix} = \sum_{i=1}^m x_i \frac{\partial f_j}{\partial x_i}(tx). \qquad \square$$

Theorem 3.12 (Homotopy invariance). *Let ω be closed. Let $\gamma, \delta \colon [0,1] \to U$ be piecewise $\mathcal{C}^1$-loops that are loop-homotopic. Then*

$$\int_\gamma \omega = \int_\delta \omega.$$

In the proof we will use the followint notation. For $x, y \in V$ let $\overline{x, y}$ be the path $[0,1] \to V$, $t \mapsto x + t(y - x)$ (the line segment from x to y). More generally set for $x_1, \dots, x_r \in V$

$$\overline{x_1, \dots, x_r} := \overline{x_1, x_2} \cdot \ \dots \ \cdot \overline{x_{r-1}, x_r}.$$

Such paths are called *piecewise affine linear.*

Proof. Choose a norm $\|\cdot\|$ on V. Let $H \colon [0,1] \times [0,1] \to U$ be a loop homotopy from γ to δ. *(i).* As $H([0,1] \times [0,1])$ is compact, there exists $\varepsilon > 0$ such that

$$\|H(s,t) - y\| \geq \varepsilon \qquad \forall\, (t,s) \in [a,b] \times [0,1], y \in V \setminus U.$$

(ii). As $[0,1] \times [0,1]$ is compact, H is uniformly continuous. Hence there exists $\delta > 0$ such that

$$(*) \qquad |t - t'| < \delta, |s - s'| < \delta \quad \Rightarrow \quad \|H(t,s) - H(t',s')\| < \frac{\varepsilon}{2}.$$

Choose $0 = t_0 < t_1 < \dots < t_m = 1$ and $0 = s_0 < s_1 < \dots < s_l = 1$ with $|t_j - t_{j-1}| < \delta$ for all j and $|s_k - s_{k-1}| < \delta$ for all k. Set $A_{j,k} := H(t_j, s_k)$ ($\Rightarrow A_{0,k} = A_{m,k}$ for all k) and define piecewise $\mathcal{C}^1$-loops

$$\gamma_k := \overline{A_{0,k}, A_{1,k}, \dots, A_{m,k}}.$$

for $0 \leq k \leq l$.
(iii). For all $1 \leq j \leq m$, $0 \leq k \leq l$ define piecewise $\mathcal{C}^1$-loops

$$\sigma_{j,k} := \overline{A_{j-1,k-1}, A_{j-1,k}, A_{j,k}, A_{j,k-1}, A_{j-1,k-1}}.$$

The image of $\sigma_{j,k}$ is contained in $B_\varepsilon(A_{j-1,k-1})$ by (*), and $B_\varepsilon(A_{j-1,k-1})$ is a convex set which is contained in U by (i). Therefore the Lemma of Poincaré (Lemma 3.11) implies that $\int_{\sigma_{j,k}} \omega = 0$. Therefore the integral of ω over the following paths are equal

$$\begin{gathered} \gamma_{k-1}, \\ \sigma_{1,k} \cdot \overline{A_{0,k-1}, A_{1,k-1}} \cdot \sigma_{2,k} \cdot \overline{A_{1,k-1}, A_{2,k-1}} \cdot \ \dots \ \cdot \sigma_{m,k} \cdot \overline{A_{m-1,k-1}, A_{m,k-1}}, \\ \overline{A_{0,k-1}, A_{0,k}} \cdot \gamma_k \cdot \overline{A_{m,k}, A_{m,k-1}}, \\ \gamma_k. \end{gathered}$$

Therefore we obtain $\int_{\gamma_0} \omega = \int_{\gamma_l} \omega$.
(iv). In the same way as in (iii) one can prove that $\int_\gamma \omega = \int_{\gamma_0} \omega$ and $\int_\delta \omega = \int_{\gamma_l} \omega$. $\square$

Theorem 3.13 (Poincaré lemma, global version). *Let $U \subseteq \mathbb{R}^m$ be open simply connected. Let $\omega\colon U \to \mathrm{Hom}_{\mathbb{R}}(V, W)$ be a continuously differentiable W-valued 1-form on U. Then*

$$\omega \textit{ closed} \Leftrightarrow \omega \textit{ exact.}$$

Proof. By Theorem 2.28 it suffices to show that for any piecewise $\mathcal{C}^1$-loops γ one has

$$(*) \qquad \int_\gamma \omega = 0.$$

But as U is simply connected, γ is null-homotopic, i.e., $\gamma \simeq \varepsilon_x$, where $x = \gamma(0)$. Hence Theorem 3.12 shows

$$\int_\gamma \omega = \int_{\varepsilon_x} \omega = 0. \qquad \square$$

Corollary 3.14. *Let $\omega\colon U \to \mathrm{Hom}(V, W)$ be closed. Let $\gamma, \delta\colon [a, b] \to U$ be homotopic piecewise $\mathcal{C}^1$-paths in U. Then*

$$\int_\gamma \omega = \int_\delta \omega.$$

Proof. $\gamma \simeq \delta \Rightarrow \gamma \cdot \delta^-$ null-homotopic. Hence:

$$\int_\gamma \omega - \int_\delta \omega = \int_{\gamma\cdot\delta^-} \omega \overset{3.12}{=} 0. \qquad \square$$

Remark and Definition 3.15. The proof of Theorem 3.12 shows that every path γ (not necessarily piecewise $\mathcal{C}^1$) in U is homotopic in U to a piecewise affine linear path $\tilde\gamma$ (which is in particular piecewise $\mathcal{C}^1$).[3] This allows us to define for every path $\gamma\colon [a, b] \to U$ and for every *closed* 1-form ω on U:

$$\int_\gamma \omega := \int_{\tilde\gamma} \omega.$$

This is independent of the choice of $\tilde\gamma$ by Corollary 3.14.

[3]Indeed, the proof shows that for every path γ there exists a piecewise affine linear path $\overline{x_0, \dots, x_k}$ with $x_0 = \gamma(0)$ and $x_k = \gamma(1)$ such that γ can be written as $\gamma_1 \cdot \dots \cdot \gamma_k$ (up to reparametrization which we may do because Remark 3.4 (1) shows that a reparametrized way is homotopic to the original one) and such that for all $i = 1, \dots, k$ the images of $\overline{x_{i-1}, x_i}$ and γ_i are contained in a convex open subset B_i which is contained in U (in fact, we may even assume that $x_i = \gamma_i(1)$ but we don't need this in the sequel). Let $\sigma_0 := \varepsilon_{x_0}$ and $\sigma_i := \overline{\gamma_i(1), x_i}$ for $i = 1, \dots, k$. Then $\sigma_k = \varepsilon_{x_k}$. The paths $\gamma_i \cdot \sigma_i$ and $\sigma_{i-1} \cdot \overline{x_{i-1}, x_i}$ are then contained in B_i and hence are homotopic. Hence induction shows that

$$\gamma_1 \cdot \dots \cdot \gamma_i \cdot \sigma_i \simeq \sigma_0 \cdot \overline{x_0, x_1} \cdot \dots \cdot \overline{x_{i-1}, x_i}$$

for all $i = 1, \dots, k$ and in particular

$$\gamma \simeq \gamma_1 \cdot \dots \cdot \gamma_k \cdot \varepsilon_{x_k} \simeq \varepsilon_{x_0} \cdot \overline{x_0, x_1} \cdot \dots \cdot \overline{x_{k-1}, x_k}.$$

(C) The fundamental group

Definition 3.16. Let X be a topological space, $x \in X$. Define

$$\pi_1(X,x) := \{\, \gamma \colon [0,1] \to X \ ; \ \gamma \text{ path with } \gamma(0) = \gamma(1) = x\}/ \simeq,$$

i.e., $\pi_1(X,x)$ is the set of homotopy classes $[\gamma]$ of closed paths γ starting (and ending) in x. Define a multiplication on $\pi_1(X,x)$ by

$$[\gamma][\delta] := [\gamma \cdot \delta].$$

By Remark 3.4 this is well-defined and yields a group structure on $\pi_1(X,x)$. The neutral element is $[\varepsilon_x]$ and the inverse of $[\gamma] \in \pi_1(X,x)$ is $[\gamma^-]$.
In the sequel we simply write γ instead of $[\gamma]$ for elements in $\pi_1(X,x)$.

Remark 3.17. Let X be a path-connected topological space, $x, y \in X$. Choose a path σ from x to y. Then

$$\varphi \colon \pi_1(X,x) \to \pi_1(X,y), \qquad \gamma \mapsto \sigma^- \cdot \gamma \cdot \sigma$$

is an isomorphism of groups: Remark 3.4 shows that φ is well defined and that for $\gamma, \gamma' \in \pi_1(X,x)$ one has

$$\varphi(\gamma)\varphi(\gamma') = \sigma^- \cdot \gamma \cdot \sigma \cdot \sigma^- \cdot \gamma' \cdot \sigma \simeq \sigma^- \cdot \gamma \cdot \varepsilon_x \cdot \gamma' \cdot \sigma \simeq \sigma^- \cdot \gamma \cdot \gamma' \cdot \sigma = \varphi(\gamma\gamma').$$

Hence φ is a group homomorphism. An inverse is given by $\delta \mapsto \sigma \cdot \delta \cdot \sigma^-$.

Example 3.18. A path-connected topological space X is simply connected if and only if $\pi_1(X,x) = 1$ for all (equivalently, for one) $x \in X$.

Example 3.19. For $n \in \mathbb{Z}$ let $\gamma_n \colon [0,1] \to \mathbb{C}^\times$, $\gamma_n(t) = \exp(2\pi i n t)$. We will see in Section 8 that

$$\mathbb{Z} \to \pi_1(\mathbb{C}^\times, 1), \qquad n \mapsto \gamma_n$$

is an isomorphism of groups.

In general, $\pi_1(X,x)$ is not an abelian group (e.g., if $X = \mathbb{R}^2 \setminus \{P,Q\}$ for $P \neq Q$ points in $\mathbb{R}^2$).

4 Holomorphic 1-forms

(A) Complex path integrals

We now consider $\mathbb{C}$-valued 1-forms on open subsets U of the 2-dimensional $\mathbb{R}$-vector space $\mathbb{C}$. As usual we denote the coordinate function $\mathrm{Re} \colon \mathbb{C} \to \mathbb{R}$ by x and $\mathrm{Im} \colon \mathbb{C} \to \mathbb{R}$ by y. Thus every $\mathbb{C}$-valued 1-form on U is of the form

$$\omega = f_x dx + f_y dy$$

for functions $f_x, f_y \colon U \to \mathbb{C}$.

Notation: Let $U \subseteq \mathbb{C}$ open, $f \colon U \to \mathbb{C}$ a map.

Remark and Definition 4.1. Let $f\colon U \to \mathbb{C}$ be continuously real differentiable. Recall that

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy. \tag{*}$$

For instance let $z\colon U \to \mathbb{C}$, $x+iy \mapsto x+iy$ and $\bar{z}\colon U \to \mathbb{C}$, $x+iy \mapsto x-iy$. Then

$$dz = dx + idy, \qquad d\bar{z} = dx - idy.$$

Hence we can rewrite (*) as follows.

$$df = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right)dz + \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right)d\bar{z}. \tag{**}$$

If we define the *Wirtinger derivatives*

$$\frac{\partial f}{\partial z} := \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right) : U \to \mathbb{C}, \qquad \frac{\partial f}{\partial \bar{z}} := \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right) : U \to \mathbb{C},$$

then (**) can be written

$$df = \frac{\partial f}{\partial z}dz + \frac{\partial f}{\partial \bar{z}}d\bar{z}.$$

Proposition 4.2. *Let $f\colon U \to \mathbb{C}$ be continuously real differentiable. The following assertions are equivalent.*
(i) *f is holomorphic.*
(ii) $\frac{\partial f}{\partial \bar{z}} = 0$.
(iii) *The 1-form $f dz$ is closed.*
In this case one has for the complex derivative

$$f' = \frac{\partial f}{\partial z}, \tag{4.2.1}$$

and hence $df = f'dz$.

A similar statement holds also for holomorphic maps $f\colon U \subseteq \mathbb{C}^m \to \mathbb{C}^n$.

Proof. "(i) $\Leftrightarrow$ (ii)": We have $\frac{\partial f}{\partial x} = \frac{\mathrm{Re}(f) + i\mathrm{Im}(f)}{\partial x}$ and similar for $\frac{\partial f}{\partial y}$. Hence:

$$\begin{aligned}\frac{\partial f}{\partial \bar{z}} = 0 \Leftrightarrow \frac{\partial f}{\partial x} &= -i\frac{\partial f}{\partial y}\\ &\Leftrightarrow \frac{\partial \mathrm{Re}(f)}{\partial x} = \frac{\partial \mathrm{Im}(f)}{\partial y} \quad \text{and} \quad \frac{\partial \mathrm{Im}(f)}{\partial x} = -\frac{\partial \mathrm{Re}(f)}{\partial y}\\ &\overset{(1.12.1)}{\Leftrightarrow} f \text{ holomorphic.}\end{aligned}$$

"(ii) $\Leftrightarrow$ (iii)": We have:

$$\begin{aligned} f dz = f dx + i f dy \text{ closed} &\Leftrightarrow \frac{\partial f}{\partial y} = i\frac{\partial f}{\partial x}\\ &\Leftrightarrow \frac{\partial f}{\partial \bar{z}} = 0.\end{aligned}$$

Finally (4.2.1) follows from (1.12.3) and (1.12.1). ☐

Definition and Remark 4.3. Let $f\colon U \to \mathbb{C}$ be holomorphic. A *holomorphic primitive of f* (German: *holomorphe Stammfunktion von f*) is a holomorphic function $F\colon U \to \mathbb{C}$ such that $F' = f$.
If $G\colon U \to \mathbb{C}$ is a second primitive of f, then $F - G\colon U \to \mathbb{C}$ is locally constant (Proposition 1.15).

Remark 4.4. Let $f\colon U \to \mathbb{C}$ be holomorphic. Then $f dz$ is exact if and only if there exists a holomorphic primitive F of f (Proposition 4.2). In this case one has $dF = f dz$ (4.2.1).

Remark 4.5. Let $\gamma\colon [a, b] \to \mathbb{C}$ be a $\mathcal{C}^1$-path ($a, b \in \mathbb{R}$, $a < b$) and let $f\colon \{\gamma\} \to \mathbb{C}$ be continuous. Then

$$\begin{aligned}\int_\gamma f\, dz &\overset{2.11}{=} \int_a^b f(\gamma(t))\mathrm{Re}(\gamma)'(t) + if(\gamma(t))\mathrm{Im}(\gamma)'(t)\, dt \\ &= \int_a^b f(\gamma(t))\gamma'(t)\, dt.\end{aligned}$$

Example 4.6. Let $z_0 \in \mathbb{C}$, let $r \in \mathbb{R}^{>0}$, $n \in \mathbb{Z}$. Let $\gamma_n\colon [0, 1] \to \mathbb{C}$, $\gamma_n(t) := z_0 + r\exp(2\pi int)$. Then we have

$$\int_{\gamma_n} \frac{1}{z - z_0}\, dz = \int_0^1 \frac{1}{r\exp(2\pi int)} 2\pi inr\exp(2\pi int)\, dt = 2\pi in.$$

In particular:

$$\int_{\partial B_r(z_0)} \frac{1}{z - z_0}\, dz = 2\pi i. \tag{4.6.1}$$

Remark 4.7. Let $f\colon U \to \mathbb{C}$ be holomorphic. As $f dz$ is a closed 1-form we can use Remark 3.15 to define

$$\int_\gamma f\, dz$$

for every path $\gamma\colon [a, b] \to U$ (not necessarily piecewise $\mathcal{C}^1$).

Definition and Remark 4.8. Let $\gamma\colon [a, b] \to U$ be a piecewise $\mathcal{C}^1$-path ($a, b \in \mathbb{R}$, $a < b$). Then

$$L(\gamma) := \int_a^b |\gamma'(t)|\, dt$$

is called the *length of γ*.

Let $f\colon \{\gamma\} \to \mathbb{C}$ be continuous. Set

$$\|f\|_\gamma := \sup_{t\in[a,b]} |f(\gamma(t))| = \sup_{z\in\{\gamma\}} |f(z)|.$$

Then

$$\begin{aligned}\left|\int_\gamma f\,dz\right| &= \left|\int_a^b f(\gamma(t))\gamma'(t)\,dt\right| \\ &\le \int_a^b |f(\gamma(t))||\gamma'(t)|\,dt \\ &\le \|f\|_\gamma L(\gamma).\end{aligned} \tag{4.8.1}$$

Remark 4.9. Assume that a holomorphic function $f\colon U \to \mathbb{C}$ has a primitive F, $\gamma\colon [a,b] \to U$ a path. Then (Proposition 2.16, Remark 3.15):

$$\int_\gamma f\,dz = F(\gamma(b)) - F(\gamma(a)).$$

Proposition 4.10. *Let $U \subseteq \mathbb{C}$ be open, let $\gamma\colon [a,b] \to U$ be a path, and let $f\colon U\times\{\gamma\} \to \mathbb{C}$ be continuous. Moreover assume that for $z \in \{\gamma\}$ the function $U \to \mathbb{C}$, $w \mapsto f(w,z)$ is holomorphic with derivative $\frac{\partial f}{\partial w}$. Then*

$$F\colon U \to \mathbb{C}, \qquad F(w) := \int_\gamma f(w,z)\,dz$$

is holomorphic with

$$F'(w) = \int_\gamma \frac{\partial f}{\partial w}(w,z)\,dz.$$

Proof. This follows from (4.2.1) because an analogous assertion has been shown in Analysis 2 for partial derivatives. □

(B) Cauchy's theorem and Cauchy integral formula

Theorem 4.11 (Cauchy's Theorem). *Let $f\colon U \to \mathbb{C}$ be holomorphic.*

(1) *Let $\gamma, \delta\colon [a,b] \to U$ be paths. Assume that γ and δ are homotopic or that γ and δ are loops which are loop-homotopic. Then*

$$\int_\gamma f\,dz = \int_\delta f\,dz.$$

(2) *f has a primitive if and only if*

$$(*) \qquad \int_\gamma f\,dz = 0 \qquad \text{for every loop } \gamma \text{ in } U.$$

(3) *Let U be simply connected. Then f has a primitive.*

Proof. Proposition 4.2 $\Rightarrow$ $f\,dz$ is closed. Hence Theorem 3.12 and Corollary 3.14 imply (1). The equivalence in (2) follows from Theorem 2.28. If U is simply connected, Corollary 3.13 implies that $f\,dz$ is exact, i.e., f has a primitive (Remark 4.4). $\square$

Remark 4.12. Let $G \subseteq \mathbb{C}$ be a domain. If $f\colon G \to \mathbb{C}$ is a holomorphic function that has a primitive (which is always the case, if G is simply connected), then such a primitive F of f can be constructed as follows. Fix $w_0 \in G$ and set

$$F\colon U \to \mathbb{C}, \qquad F(w) := \int_{w_0}^{w} f\,dz,$$

where $\int_{w_0}^{w}$ denotes the path integral over some path with startpoint w_0 and endpoint w. This follows from the proof of Theorem 2.28.

Remark 4.13.

(1) In general there exist holomorphic functions which do not have a primitive. For instance $f\colon \mathbb{C}^\times \to \mathbb{C}$, $f(z) = 1/z$ has no primitive by (4.6.1). But it has a primitive on every open simply connected subspace of $\mathbb{C}^\times$ (see the definition of logarithms in Section 6).

(2) Cauchy's theorem shows in particular that the restriction of f to small discs always has a primitive. Hence primitives of f always exist locally.

(3) Even if U is not simply connected, then there exist holomorphic functions $f\colon U \to \mathbb{C}$ that have a primitive (e.g., $f\colon \mathbb{C}^\times \to \mathbb{C}$, $z \mapsto z^n$ has for $-1 \neq n \in \mathbb{Z}$ the primitive $z \mapsto \frac{1}{n+1} z^{n+1}$).

Theorem 4.14 (Cauchy integral formula, local version). *Let $U \subseteq \mathbb{C}$ be open, $\tilde{z} \in U$ and $r > 0$ such that $\overline{B_r(\tilde{z})} \subseteq U$. Let $f\colon U \to \mathbb{C}$ be holomorphic. Then for every $z_0 \in B_r(\tilde{z})$ we have*

$$f(z_0) = \frac{1}{2\pi i} \int_{\partial B_r(\tilde{z})} \frac{f(z)}{z - z_0}\,dz$$

Proof. Let $\varepsilon > 0$ with $B_\varepsilon(z_0) \subseteq B_r(\tilde{z})$. By Example 3.5, $\partial B_\varepsilon(z_0)$ and $\partial B_r(\tilde{z})$ are loop homotopic in $\mathbb{C} \setminus \{z_0\}$. Hence we obtain

$$\begin{aligned}
&\int_{\partial B_r(\tilde{z})} \frac{f(z)}{z - z_0}\,dz - 2\pi i f(z_0) \\
\overset{4.6,4.11}{=}\ &\int_{\partial B_\varepsilon(z_0)} \frac{f(z)}{z - z_0}\,dz - f(z_0) \int_{\partial B_\varepsilon(z_0)} \frac{1}{z - z_0}\,dz \\
=\ &\int_{\partial B_\varepsilon(z_0)} \frac{f(z) - f(z_0)}{z - z_0}\,dz \\
\overset{\varepsilon \to 0}{\longrightarrow}\ &0,
\end{aligned}$$

where the last line holds because $|\frac{f(z)-f(z_0)}{z-z_0}|$ is bounded in a neighborhood of z_0 (f is holomorphic in z_0) and $L(\partial B_\varepsilon(z_0)) = 2\pi\varepsilon \to 0$ for $\varepsilon \to 0$. $\square$

5 Properties of holomorphic functions I

Notation: U is always an open subspace of $\mathbb{C}$.

(A) Analytic functions

Remark and Definition 5.1. Let $z_0 \in \mathbb{C}$. Recall that a complex power series in z_0 is a series of the form

$$(*) \qquad \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

with $z \in \mathbb{C}$ and $a_n \in \mathbb{C}$ for all $n \in \mathbb{N}_0$. We usually consider (*) as sequence of functions $(f_N)_{N \in \mathbb{N}_0}$ where

$$f_N \colon \mathbb{C} \to \mathbb{C}, \qquad z \mapsto \sum_{n=0}^{N} a_n (z - z_0)^n.$$

Then

$$(5.1.1) \qquad \rho := \frac{1}{\limsup_n |a_n|^{1/n}} \in \overline{\mathbb{R}}^{\geq 0}.$$

is called the *radius of convergence of (*)*. We have:
(1) For $z \in \mathbb{C}$ with $|z - z_0| > \rho$ the power series (*) does not converge.
(2) The power series (*) converges absolutely and locally uniformly on $B_\rho(z_0)$ (more precisely: the series (*) converges absolutely for all $z \in B_\rho(z_0)$, and the sequence $(f_N)_{N \in \mathbb{N}_0}$ converges locally uniformly on $B_\rho(z_0)$).
The power series (*) is called *convergent*, if $\rho > 0$.

Definition 5.2. Let $f \colon U \to \mathbb{C}$ be a map.
(1) Let $z_0 \in U$. Then f is called *analytic in* z_0, if there exists a convergent power series $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ such that $f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$ for all z in a neighborhood of z_0.
(2) f is called *analytic* if f is analytic in z_0 for all $z_0 \in U$.

Proposition 5.3. *Let $f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$ be a power series, $\rho > 0$ its radius of convergence. Then f is an analytic function on $B_\rho(z_0)$.*

Proof. See Analysis 1. □

Remark 5.4. If f is analytic on an open subset U, it does *not* mean that there exists a power series $\sum_n a_n (z - z_0)^n$ such that $f(z) = \sum_n a_n (z - z_0)^n$ for all $z \in U$. The function f can only locally expressed as a power series.
For instance, consider $f \colon \mathbb{C}^\times \to \mathbb{C}$, $f(z) := 1/z$. If there existed a power series $\sum_n a_n (z - z_0)^n$ such that $f(z) = \sum_n a_n (z - z_0)^n$ for all $z \in \mathbb{C}^\times$, then this power series would have to converge on a circle B whose closure contains $\mathbb{C}^\times$. Hence B would be $\mathbb{C}$. But then f could be extended continuously (even analytically) to the function $\tilde{f} \colon \mathbb{C} \to \mathbb{C}$, $\tilde{f}(z) = \sum_n a_n (z - z_0)^n$, which is not possible because $\lim_{z \to 0} f(z)$ does not exist in $\mathbb{C}$.

Verbatim as in Analysis 1 one shows the following proposition.

Proposition 5.5. *Let $f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n$ be a power series, ρ its radius of convergence.*

(1) *The power series $\sum_{n=0}^{\infty} na_n(z-z_0)^{n-1}$ has also ρ as radius of convergence.*

(2) *The function $f\colon B_\rho(z_0) \to \mathbb{C}$ is infinitely often complex differentiable (in particular holomorphic) and*
$$f'(z) = \sum_{n=1}^{\infty} na_n(z-z_0)^{n-1}$$
for $z \in B_\rho(z_0)$.

(3)
$$a_n = \frac{f^{(n)}(z_0)}{n!}.$$

Example 5.6. (1) The power series $\exp(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}$ has radius of convergent ∞. Hence
$$\exp\colon \mathbb{C} \to \mathbb{C}$$
is a holomorphic function, called the *exponentional function*. We have
$$\exp'(z) \overset{5.5}{=} \sum_{n=1}^{\infty} \underbrace{\frac{n}{n!}}_{=\frac{1}{(n-1)!}} z^{n-1} = \sum_{n=0}^{\infty} \frac{1}{n!} z^n = \exp(z).$$

(2) The power series
$$\cos(z) := \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} z^{2n},$$
$$\sin(z) := \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1}$$
have radius of convergence ∞ and therefore define holomorphic functions $\cos\colon \mathbb{C} \to \mathbb{C}$ and $\sin\colon \mathbb{C} \to \mathbb{C}$. One has
$$\sin'(z) = \cos(z) \qquad \text{and} \qquad \cos'(z) = -\sin(z)$$
for all $z \in \mathbb{C}$.

(B) Holomorphic functions are analytic

Lemma 5.7. *Let $\gamma\colon [a,b] \to \mathbb{C}$ be a path, let $g\colon \{\gamma\} \to \mathbb{C}$ be continuous. Assume one of the following hyoptheses.*

(a) *The path γ is piecewise $\mathcal{C}^1$.*

(b) *There exists an open neighborhood W of $\{\gamma\}$ and a holomorphic function $\tilde{g}\colon W \to \mathbb{C}$ such that $\tilde{g}_{|\{\gamma\}} = g$.*

Define

$$f\colon \mathbb{C}\setminus\{\gamma\} \to \mathbb{C}, \qquad f(w) := \int_\gamma \frac{g(z)}{z-w}\,dz.$$

For $z_0 \in \mathbb{C}\setminus\{\gamma\}$ *we set* $\rho := \operatorname{dist}(z_0, \{\gamma\}) := \inf_{z\in\{\gamma\}} d(z, z_0)$. *Then the function* $f_{|B_\rho(z_0)}$ *has a power series expansion in* z_0 *with*

$$f^{(n)}(z_0) = n! \int_\gamma \frac{g(z)}{(z-z_0)^{n+1}}\,dz. \tag{5.7.1}$$

Proof. As $\{\gamma\}$ is compact, we have $\rho > 0$.
We claim that for all $0 < r < \rho$ the restriction $f_{|B_r(z_0)}$ has a power series expansion in z_0 (then the claim shows that the power series expansion has radius of convergence $\geq r$ for all $r < \rho$ and hence it radius of convergence is $\geq \rho$).
Under the assumption (b) we may replace γ by a piecewise $\mathcal{C}^1$-path that is homotopic to γ in $W \setminus B_r(z_0)$. This does not change $f_{|B_r(z_0)}$ by Cauchy's theorem 4.11. Thus we may assume that γ is piecewise $\mathcal{C}^1$, i.e. we are in case (a).
For $z \in \{\gamma\}$ and $w \in B_r(z_0)$ we have $|w - z_0| < |z - z_0|$. Hence we may write

$$\begin{aligned}\frac{1}{z-w} &= \frac{1}{z-z_0}\frac{1}{1-\frac{w-z_0}{z-z_0}}\\ &= \frac{1}{z-z_0}\sum_{n=0}^{\infty}\left(\frac{w-z_0}{z-z_0}\right)^n\\ &= \sum_{n=0}^{\infty}\frac{(w-z_0)^n}{(z-z_0)^{n+1}}\end{aligned}$$

and the series converges locally uniformly as a function in z. Moreover, g is bounded on $\{\gamma\}$ and therefore $\sum_{n=0}^\infty g(z)\frac{(w-z_0)^n}{(z-z_0)^{n+1}}$ converges also locally uniformly. Hence we obtain

$$\begin{aligned}f(w) &= \int_\gamma \sum_{n=0}^{\infty} g(z)\frac{(w-z_0)^n}{(z-z_0)^{n+1}}\,dz\\ &\overset{2.20}{=} \sum_{n=0}^{\infty} \underbrace{\left(\int_\gamma \frac{g(z)}{(z-z_0)^{n+1}}\,dz\right)}_{=:a_n}(w-z_0)^n\end{aligned}$$

and f has a power series expansion in z_0. By Proposition 5.5 we then know that

$$f^{(n)}(z_0) = n!a_n.$$

□

Theorem 5.8. *let* $f\colon U \to \mathbb{C}$ *be holomorphic,* $z_0 \in U$*, and let* $R \in \overline{\mathbb{R}}^{>0}$ *such that* $\overline{B_R(z_0)} \subseteq U$. *Then* $f_{|B_R(z_0)}$ *has a power series expansion*

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$$

such that for all $n \in \mathbb{N}_0$ one has

$$a_n = \frac{f^{(n)}(z_0)}{n!} = \frac{1}{2\pi i} \int\limits_{\partial B_r(\tilde{z})} \frac{f(z)}{(z-z_0)^{n+1}}\, dz, \tag{5.8.1}$$

where $B_r(\tilde{z})$ is any disc such that $z_0 \in B_r(\tilde{z})$ and $\overline{B_r(\tilde{z})} \subseteq U$. Furthermore one has

$$|a_n| \leq \frac{\|f\|_{\partial B_r(\tilde{z})}}{r^n}. \tag{5.8.2}$$

Proof. By the Cauchy Integral formula (Theorem 4.14) one has

$$f(w) = \frac{1}{2\pi i} \int\limits_{\partial B_r(\tilde{z})} \frac{f(z)}{z-w}\, dz$$

for all $w \in B_r(\tilde{z})$. Choosing $r = R$ and $\tilde{z} = z_0$ and applying Lemma 5.7 to $\gamma = \partial B_R(z_0)$ shows that f has a power series expansion in z_0 on $B_R(z_0)$. Applying (5.7.1) to $\gamma = \partial B_r(\tilde{z})$ we obtain (5.8.1).
It remains to show (5.8.2): Let $\gamma := \partial B_r(\tilde{z})$. Then

$$|a_n| \leq \frac{1}{2\pi} \left\| \frac{f(z)}{(z-\tilde{z})^{n+1}} \right\|_\gamma L(\gamma) = \frac{1}{2\pi} \|f(z)\|_\gamma \frac{1}{r^{n+1}} r 2\pi = \frac{\|f\|_\gamma}{r^n}. \qquad \square$$

Corollary 5.9. *$U \subseteq \mathbb{C}$ open, $f\colon U \to \mathbb{C}$ function. Then the following assertions are equivalent:*
(i) *f is holomorphic.*
(ii) *f is infinitely often complex differentiable and all higher derivatives $f^{(n)}$ are holomorphic.*
(iii) *f is analytic.*

Proof. "(iii) ⇒ (ii)": Proposition 5.5. "(ii) ⇒ (i)": Clear.
"(i) ⇒ (iii)": Theorem 5.8 $\square$

Corollary 5.10. *Let $f\colon U \to \mathbb{C}$ be holomorphic, $z_0 \in U$, $\sum_n a_n(z-z_0)^n$ its power series expansion in z_0, and let ρ be its radius of convergence. Then*

$$\rho = \sup\{\, r \in \overline{\mathbb{R}}^{>0} \;;\; \exists \textit{ holomorphic function } \tilde{f}\colon U \cup B_r(z_0) \textit{ with } \tilde{f}_{|U} = f\}.$$

Proof. Let ρ' be the right hand side. As a power series is holomorphic within its open disc of convergense, an extension $\tilde{f}$ of f to $U \cup B_\rho(z_0)$ exists. Hence $\rho \leq \rho'$.
Conversely, let us show that $\rho \geq \rho'$. For all $0 < R < \rho'$ we apply (5.8.2) to $\tilde{f}\colon U \cup B_{\rho'}(z_0) \to \mathbb{C}$. As f and $\tilde{f}$ are equal in a neighborhood of z_0, we have $f^{(n)}(z_0) = \tilde{f}^{(n)}(z_0)$. Hence f and $\tilde{f}$ have the same power series expansion $\sum_{n=0}^\infty a_n(z-z_0)^n$ in z_0. Set $C := \|\tilde{f}\|_{\partial B_R(z_0)}$. Then (5.8.2) shows

$$|a_n|^{1/n} \leq C^{1/n}/R. \tag{*}$$

If $C = 0$ this implies $a_n = 0$ for all n and hence $\rho = \infty$ and in particular $\rho \geq \rho'$. Thus we may assume that $C > 0$. Then (*) shows that for all $R < \rho'$ one has

$$\rho = \frac{1}{\limsup_n |a_n|^{1/n}} \geq R$$

because $\lim_n C^{1/n} = 1$. Hence $\rho \geq \rho'$. □

Example 5.11. Let $f\colon \mathbb{C} \setminus \{1\} \to \mathbb{C}$, $f(z) := \exp(z)/(z-1)$. Then the radius of convergence of the power series for f at 0 (resp. at i) is 1 (resp. $\sqrt{2}$).
The radius of convergence of the power series for $f\colon \mathbb{C} \setminus \{0\} \to \mathbb{C}$, $z \mapsto \frac{\exp(z)-1}{z}$ at any $z_0 \in \mathbb{C} \setminus \{0\}$ is ∞ (we will see that f can be extended holomorphically to 0).

Corollary 5.12. *Let $U \subseteq \mathbb{C}$ be open and let $f\colon U \to \mathbb{C}$ be continuous such that*

$$\int_\gamma f\,dz = 0$$

for every piecewise $\mathcal{C}^1$-loop γ in U. Then f is holomorphic.

In fact, it suffices to take triangular loops γ (see Exercise).

Proof. Theorem 2.28 $\Rightarrow \exists\, F\colon U \to \mathbb{C}$ a $\mathcal{C}^1$-map with $dF = f dz$. In particular $\frac{\partial F}{\partial \bar{z}} = 0$. Hence F is holomorphic with $F' = f$ by Proposition 4.2. Therefore f is holomorphic by Theorem 5.9. □

Theorem 5.13 (Riemann extension theorem). *Let $U \subseteq \mathbb{C}$ be open, $z_0 \in U$. Let $f\colon U \setminus \{z_0\} \to \mathbb{C}$ be holomorphic and bounded near z_0 (i.e., $\exists\, \varepsilon > 0$, $C \in \mathbb{R}^{\geq 0}$ such that $|f(z)| \leq C$ for all $z \in B_\varepsilon(z_0) \setminus \{z_0\}$). Then there exists a unique holomorphic function $\tilde{f}\colon U \to \mathbb{C}$ with $\tilde{f}_{|U\setminus\{z_0\}} = f$.*

Proof. The uniqueness of $\tilde{f}$ is clear because $U \setminus \{z_0\}$ is dense in U. We show the existence. After translation we may assume that $z_0 = 0$ (to simplify the notation).
As f is bounded near $z_0 = 0$, the function

$$F\colon U \to \mathbb{C}, \qquad F(z) := \begin{cases} f(z)z, & z \neq 0; \\ 0, & z = 0; \end{cases}$$

is holomorphic on $U \setminus \{0\}$ and continuous on U.
(i). Let us first assume that there exists a continuous extension $\tilde{f}$ of f. Applying Exercise 18(b)[4] to $\tilde{f}$, we see that $\tilde{f}$ is holomorphic. Hence the theorem is proved under the above additional assumption.
(ii). We can now apply (ii) to the function F and see that F is holomorphic. In particular $F'(0) = \lim_{z\to 0} f(z)$ exists. Hence f can be extended to a continuous function $\tilde{f}$ on U. Using again (ii) we see that $\tilde{f}$ is holomorphic. □

[4]**Aufgabe 18(b)**: Let $U \subseteq \mathbb{C}$ be open, $f\colon U \to \mathbb{C}$ continuous such that $f_{|U\setminus\mathbb{R}}$ is holomorphic. Then f is holomorphic.

(C) Uniform limits of holomorphic functions

Theorem 5.14 (Weierstraß' theorem of convergence). *Let $U \subseteq \mathbb{C}$ be open, $(f_k\colon U \to \mathbb{C})_{k\in\mathbb{N}}$ a locally uniformly convergent sequence of holomorphic functions. Set $f := \lim_{k\to\infty} f_k$.*
(1) *f is holomorphic.*
(2) *For all $n \in \mathbb{N}$, the sequence of n-th derivatives $(f_k^{(n)})_k$ converges locally uniformly to $f^{(n)}$.*

Proof. It suffices to show that for all $z_0 \in U$ there exists an $\varepsilon > 0$ such that $f_{|B_\varepsilon(z_0)}$ is holomorphic and such that $(f_k^{(n)}{}_{|B_\varepsilon(z_0)})_k$ converges locally uniformly to $f^{(n)}{}_{|B_\varepsilon(z_0)}$. Thus we may assume that U is convex and in particular simply connected.
(1). We know that f is continuous by Analysis 2. By Lemma 5.12 it suffices to show that $\int_\gamma f\,dz = 0$ for every loop γ in U. As f_k is holomorphic and as U is simply connected, we have $\int_\gamma f_k\,dz = 0$ for all loops γ and all k (Cauchy's theorem 4.11). As $\int_\gamma(\cdot)$ commutes with locally uniform limit (Proposition 2.20), this implies $\int_\gamma f\,dz = 0$ for every loop γ in U.
(2). Let $z_0 \in U$. Choose $R > 0$ so small such that $\overline{B_R(z_0)} \subseteq U$ and such that f_k converges uniformly to f on $\overline{B_R(z_0)}$ (possible because $f_k \to f$ locally uniformly). Let $\varepsilon > 0$. Then there exists $k_0 \in \mathbb{N}$ such that $|f_k(z) - f(z)| \leq \varepsilon$ for all $z \in \overline{B_R(z_0)}$ and for all $k \geq k_0$. For all $w \in B_{R/2}(z_0)$ we then have $|w - z| \geq R/2$ for all $z \in \partial B_R(z_0)$. Therefore we find for $w \in B_{R/2}(z_0)$ and for all $k \geq k_0$:

$$\begin{aligned}
|f^{(n)}(w) - f_k^{(n)}(w)| &= |(f - f_k)^{(n)}(w)| \\
&\overset{(5.8.1)}{=} \Big|\frac{n!}{2\pi i} \int\limits_{\partial B_R(z_0)} \frac{f(z) - f_k(z)}{(z-w)^{n+1}}\,dz\Big| \\
&\leq \frac{n!}{2\pi}\frac{\varepsilon}{(R/2)^{n+1}} R2\pi = \varepsilon C,
\end{aligned}$$

where C is a constant not depending on ε, k, or on w. This shows that $(f_k^{(n)})_k$ converges locally uniformly to $f^{(n)}$. □

Remark 5.15. Note that this is totally different from the real setting. There Weierstraß approximation theorem tells us that every continuous function on $[a, b]$ is the uniform limit of polynomial functions ($\Rightarrow$ uniform limits of polynomials are not necessarily differentiable).
Nevertheless one can ask whether every holomorphic function f is a uniform limit of polynomials. More precisely: Let $K \subset \mathbb{C}$ be compact.
Question: Does there exist for every f that is holomorphic on some open neighborhood of K a sequence of polynomial functions converging uniformly on K to f?
In general this is certainly not true: As polynomials p are holomorphic functions on $\mathbb{C}$ and $\mathbb{C}$ is simply connected, we have $\int_\gamma p\,dz = 0$ for every loop γ. As $\int_\gamma$ commutes with uniform limit, we also should have $\int_\gamma f\,dz = 0$ for every loop in K. Thus we at least need K to be simply connected. In fact one can show that in this case the

above question has indeed a positive answer (Runge's theorem, see e.g. John Conway: *Functions of One Complex Variable*, 2nd edition, Springer (1978), Chap. VIII).

(D) Liouville's theorem

Definition 5.16. A holomorphic function on $\mathbb{C}$ is called *entire* (German: *ganz*).

Corollary 5.17. *Let $c, M \in \mathbb{R}^{\geq 0}$ and let f be an entire function. Assume that*

$$\|f\|_{\partial B_R(0)} \leq MR^c$$

for R large enough (i.e., $\exists\ R_0 \in \mathbb{R}^{>0}$ such that $\|f\|_{\partial B_R(0)} \leq MR^c$ for all $R \geq R_0$). Then f is a polynomial of degree $\leq c$.

Proof. Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be the power series expansion of f on $z_0 = 0$. By (5.8.2) we have for all $n > c$

$$|a_n| \leq \frac{MR^c}{R^n} \xrightarrow{R\to\infty} 0.$$

$\square$

Corollary 5.18 (Liouville). *A bounded entire function is constant.*

Proof. Corollary 5.17 with $c = 0$. $\square$

Corollary 5.19 (Main Theorem of Algebra). $\mathbb{C}$ *is algebraically closed.*

Proof. Let $p\colon \mathbb{C} \to \mathbb{C}$, $p(z) := a_n z^n + \dots + a_1 z + a_0$, $a_i \in \mathbb{C}$, $a_n \neq 0$ be a non-constant polynomial function (i.e., $n \geq 1$). We have to show that p has a zero. Assume that $p(z) \neq 0$ for all $z \in \mathbb{C}$. Then $f = 1/p$ is an entire function. Moreover, for every sequence $(z_n)_n$ in $\mathbb{C}$ with $\lim_n |z_n| = \infty$ one has $\lim_n |p(z_n)| = |a_n||z_n| = \infty$. Hence f is bounded. Contradiction to Corollary 5.18. $\square$

(E) Identity theorem

Lemma 5.20. *Let $f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$ be a non-zero power series with radius of convergence $\rho > 0$. Let $m := \operatorname{ord}_{z_0}(f) := \inf\{\, n \in \mathbb{N}_0\ ;\ a_n \neq 0\,\}$.*
(1) *Then $f(z) = a_m(z - z_0)^m(1 + h(z))$, where $h(z)$ is a power series of the form $h(z) = b_1(z - z_0) + b_2(z - z_0)^2 + \dots$ with radius of convergence ρ.*
(2) *There exists $0 < r \leq \rho$ such that $f(z) \neq 0$ for all $z \in B_r(z_0) \setminus \{z_0\}$.*

Proof. (1).

$$\begin{aligned} f(z) &= a_m(z - z_0)^m + a_{m+1}(z - z_0)^{m+1} + \dots \\ &= a_m(z - z_0)^m(1 + b_1(z - z_0) + b_2(z - z_0)^2 + \dots). \end{aligned}$$

with $b_k = a_{m+k}/a_m$ for $k \in \mathbb{N}$. Then $\limsup_k |b_k|^{1/k} = \limsup_n |a_n|^{1/n}$ (because $\lim_k |a_m|^{1/k} = 1$). This shows (1).
(2). As $h\colon B_\rho(z_0) \to \mathbb{C}$ is continuous and $h(z_0) = 0$, there exists $0 < r \leq \rho$ such that $|h(z)| < 1$ for $z \in B_r(z_0)$ and hence $f(z) = a_m(z - z_0)^m(1 + h(z)) \neq 0$ for $z \neq z_0$. $\square$

Proposition 5.21. *Let $U \subseteq \mathbb{C}$ be open, $f\colon U \to \mathbb{C}$ holomorphic. Then*

$$\operatorname{supp}(f) := \overline{\{\, z \in U \ ; \ f(z) \neq 0 \,\}} \qquad \textit{(closure in } U\textit{!)}$$

is open and closed in U.

Proof. We only have to show that $\operatorname{supp}(f)$ is open in U. Let $z_0 \in \operatorname{supp}(f)$. Then there exists no $z_0 \in W \subseteq U$ open with $f_{|W} = 0$. Thus Lemma 5.20 (2) shows that there exists $r > 0$ with $f(z) \neq 0$ for $z \in B_r(z_0) \setminus \{z_0\}$ and hence $B_r(z_0) \subseteq \operatorname{supp}(f)$. □

Definition 5.22. Let X be a topological space. A subspace S of X is called *discrete*, if the induced topology is the discrete topology ($\Leftrightarrow \forall\, s \in S\ \exists\, s \in U \subseteq X$ open such that $U \cap S = \{s\}$).

Sometimes "discrete" is defined differently. With this definition the subspace $\{\, 1/n \ ; \ n \in \mathbb{N} \,\}$ of $\mathbb{C}$ is discrete.

Proposition 5.23. *Let $G \subseteq \mathbb{C}$ be a domain. Let $f\colon G \to \mathbb{C}$ be holomorphic with $f \neq 0$. Then $\{\, z \in G \ ; \ f(z) = 0 \,\}$ is a discrete subspace of G.*

Proof. Let $z_0 \in G$ with $f(z_0) = 0$. As f is analytic, there exists by Lemma 5.20 (2) either $r > 0$ such that $f(z) \neq 0$ for $z \in B_r(z_0) \setminus \{z_0\}$, or $f = 0$ in some neighborhood of z_0.
Assume the latter case occurs, i.e. $z_0 \notin \operatorname{supp}(f)$. As G is connected, Proposition 5.21 shows that $\operatorname{supp}(f) = \emptyset$ and hence $f = 0$; contradiction. □

Theorem 5.24 (Identity Theorem). *Let $G \subseteq \mathbb{C}$ be a domain. Let $f, g\colon G \to \mathbb{C}$ be holomorphic. Then the following assertions are equivalent.*
(i) $f = g$.
(ii) *There exists a non-discrete subspace $S \subseteq G$ such that $f(z) = g(z)$ for all $z \in S$.*
(iii) *There exists a point $z_0 \in G$ such that $f^{(n)}(z_0) = g^{(n)}(z_0)$ for all $n \in \mathbb{N}_0$ (in other words: f and g have the same power series expansion in z_0).*

Proof. "(i) $\Rightarrow$ (iii)": clear.
"(iii) $\Rightarrow$ (ii)": (iii) implies that f and g have the same power series expansion in z_0 and hence are equal on some neighborhood of z_0.
"(ii) $\Rightarrow$ (i)": Apply Proposition 5.23 to $f - g$. □

6 Special functions

(A) Extension of real analytic functions

Corollary 6.1. *Let $G \subseteq \mathbb{C}$ be a domain with $G \cap \mathbb{R} \neq \emptyset$. Let $f, g\colon G \to \mathbb{C}$ be holomorphic with $f_{|G\cap\mathbb{R}} = g_{|G\cap\mathbb{R}}$. Then $f = g$.*

Proof. $G \cap \mathbb{R}$ is non-discrete. □

Remark 6.2 (Exponential function).

(1) The exponential function $\exp\colon \mathbb{C} \to \mathbb{C}$ is the unique holomorphic function $f\colon \mathbb{C} \to \mathbb{C}$ such that $f(x) = e^x$ for all $x \in \mathbb{R}$.
(2) For $z, w \in \mathbb{C}$ one has

$$\exp(z + w) = \exp(z)\exp(w).$$

Hence $\exp(z) \neq 0$ for all $z \in \mathbb{C}$ and $\exp\colon (\mathbb{C}, +) \to (\mathbb{C}^\times, \cdot)$ is a group homomorphism.
(3) $\exp\colon \mathbb{C} \to \mathbb{C}^\times$ is surjective: For all $z = r\exp(i\varphi) \in \mathbb{C}^\times$ ($r \in \mathbb{R}^{>0}$, $\varphi \in \mathbb{R}$) there exists $w \in \mathbb{C}$ with $\exp(w) = z$ (take $w = s + i\varphi$ with $s \in \mathbb{R}$ such that $\exp(s) = r$).
(4) Let $z, \tilde{z} \in \mathbb{C}$. Then $\exp(z) = \exp(\tilde{z}) \Leftrightarrow \exists\, k \in \mathbb{Z}$ with $\tilde{z} = 2\pi ik + z$. In particular:

$$\ker(\exp) := \{\, z \in \mathbb{C} \;;\; \exp(z) = 1 \,\} = 2\pi i\mathbb{Z} := \{\, 2\pi ik \;;\; k \in \mathbb{Z} \,\}.$$

Proof. Write $z = x + iy$, $\tilde{z} = \tilde{x} + i\tilde{y}$ with $x, y, \tilde{x}, \tilde{y} \in \mathbb{R}$. Then

$$\begin{aligned} \exists\, k \in \mathbb{Z} : \tilde{z} = 2\pi ik + z &\Rightarrow \exp(\tilde{z}) = \exp(2\pi ik)\exp(z) = \exp(z) \\ &\Rightarrow \exp(\tilde{x})\exp(i\tilde{y}) = \exp(x)\exp(iy) \\ &\overset{\text{polar decomp.}}{\Rightarrow} \exp(\tilde{x}) = \exp(x), \exists\, k \in \mathbb{Z} : \tilde{y} = 2\pi k + y \\ &\Rightarrow \exists\, k \in \mathbb{Z} : \tilde{z} = 2\pi ik + z. \end{aligned}$$

□

Hence exp yields an isomorphism of groups $\mathbb{C}/(2\pi i\mathbb{Z}) \xrightarrow{\sim} \mathbb{C}^\times$.

Remark 6.3 (Sine and Cosine). The holomorphic functions $z \mapsto \sin^2(z) + \cos^2(z)$ and $z \mapsto 1$ on $\mathbb{C}$ are equal on $\mathbb{R}$ $\overset{6.1}{\Rightarrow}$ $\sin^2(z) + \cos^2(z) = 1$ for all $z \in \mathbb{C}$.

(B) Logarithm

Proposition and Definition 6.4. *Let $G \subseteq \mathbb{C}$ be open and simply connected and let $f\colon G \to \mathbb{C}$ be a holomorphic function with $f(z) \neq 0$ for all $z \in G$.*
(1) *There exists a holomorphic function $L_f\colon G \to \mathbb{C}$ such that $\exp \circ L_f = f$. Such an L_f is called a* branch of the logarithm of f on G.
(2) *If L_f and $\tilde{L}_f$ are two branches of the logarithm of f on G, then there exists $k \in \mathbb{Z}$ such that $\tilde{L}_f(z) = L_f(z) + 2\pi ik$ for all $z \in G$.*
(3) *For every branch L_f of the logarithm of f one has for $z \in G$:*

$$L_f'(z) = \frac{f'(z)}{f(z)}.$$

Proof. (3). If L_f is a branch of the logarithm of f, we have for $z \in G$

$$f'(z) = \exp'(L_f(z))L_f'(z) = f(z)L_f'(z).$$

(1). The function $G \to \mathbb{C}$, $z \mapsto f'(z)/f(z)$ is holomorphic. Hence there exists a primitive $L_f\colon G \to \mathbb{C}$ by Cauchy's theorem 4.11. Then

$$\left(\frac{f(z)}{\exp(L_f(z))}\right)' = \frac{f'(z)\exp(L_f(z)) - f(z)L_f'(z)\exp(L_f(z))}{\exp(L_f(z))^2} = 0.$$

Hence there exists $\alpha \in \mathbb{C}^\times$ such that $\exp \circ L_f = \alpha f$. Replacing L_f by $L_f + \beta$, where $\beta \in \mathbb{C}$ with $\exp(\beta) = \alpha^{-1}$, we obtain a branch of the logarithm of f.
(2). For $z \in G$ we have $\exp(\tilde{L}_f(z) - L_f(z)) = f(z)/f(z) = 1$. Thus $\tilde{L}_f - L_f$ is a continuous (even holomorphic) function $G \to S := \{ 2\pi i k \ ; \ k \in \mathbb{Z} \} \subseteq \mathbb{C}$ by Remark 6.2 (4). As S is discrete and G is connected, $\tilde{L}_f - L_f$ is constant. □

Applying this proposition to $f(z) = z$ we obtain:

Corollary 6.5. *Let $G \subseteq \mathbb{C}$ be open and simply connected with $0 \notin G$.*
(1) *There exists a function $L\colon G \to \mathbb{C}$ (called* a branch of the logarithm*) such that $\exp(L(z)) = z$ for all $z \in G$.*
(2) *If $L, \tilde{L}\colon G \to \mathbb{C}$ are two branches of the logarithm, then there exists $k \in \mathbb{Z}$ with $\tilde{L}(z) = L(z) + 2\pi i k$ for all $z \in G$.*
(3) *One has $L'(z) = 1/z$ for all branches L of the logarithm.*

Remark and Definition 6.6. Let $G \subseteq \mathbb{C}$ open and simply connected with $0 \notin G$. Assume that $G \cap \mathbb{R}^{>0}$ is a non-empty interval. Then there exists a unique branch of the logarithm $G \to \mathbb{C}$ such that its restriction to $G \cap \mathbb{R}^{>0}$ is the logarithm log defined in Analysis 1.
This branch is called the *principal branch of logarithm on G*.

Proof. The uniqueness follows from Corollary 6.1. Choose $w_0 \in G \cap \mathbb{R}^{>0}$ and define a primitive of $1/z$ (Remark 4.12) by

$$L\colon G \to \mathbb{C}, \qquad w \mapsto \int_{w_0}^{w} \frac{1}{z}\, dz - \log(w_0),$$

where $\int_{w_0}^{w}$ denotes the path integral over some path in G with startpoint w_0 and endpoint w (well defined because G is simply connected and hence all such paths are homotopic). For $x \in G \cap \mathbb{R}^{>0}$ we may take the line segment from w_0 to x (because $G \cap \mathbb{R}^{>0}$ is an interval) and obtain indeed the one-dimensional integral $L(x) = \int_{w_0}^{x} 1/t\, dt - \log(w_0)$ and hence the usual logarithm. Moreover we have $\exp(L(x)) = x$ for all $x \in G \cap \mathbb{R}^{>0}$ and hence $\exp(L(z)) = z$ for all $z \in G$ by the identity theorem. Thus L is indeed a branch of the logarithm. □

Example 6.7. It is standard to consider

$$G := \mathbb{C} \setminus \{ z \in \mathbb{R} \ ; \ z \leq 0 \} = \{ re^{i\varphi} \in \mathbb{C} \ ; \ r \in \mathbb{R}^{>0}, -\pi < \varphi < \pi \}.$$

Then the principal branch of logarithm on G is given by

$$\log\colon G \to \mathbb{C}, \qquad re^{i\varphi} \mapsto \underbrace{\log(r)}_{\text{usual real log}} + i\varphi. \tag{6.7.1}$$

(C) Powers and roots

Notation: In this subsection let $G \subseteq \mathbb{C}$ be open and simply connected, $f\colon G \to \mathbb{C}$ holomorphic with $f(z) \neq 0$ for all $z \in G$.

Definition 6.8. Let $q \in \mathbb{C}$. A holomorphic function of the form

$$G \to \mathbb{C}, z \mapsto \exp(qL_f(z)),$$

where $L_f\colon G \to \mathbb{C}$ is a branch of the logarithm of f, is called *a q-th power of f*.

It is easy to see that if h is a q-th power of f ($q \in \mathbb{C}^\times$), then f a $(1/q)$-th power of h.

Remark 6.9. Let $n \in \mathbb{N}$ and let $q = 1/n$. Then a q-th power of f is also called an *n-th root of f*.
Let $r_{f,n}, \tilde{r}_{f,n}\colon G \to \mathbb{C}$ be two n-th roots of f.

(1) There exists $k \in \{0, \dots, n-1\}$ such that

$$\tilde{r}_{f,n}(z) = \exp(\frac{2\pi ik}{n})r_{f,n}(z)$$

for all $z \in G$ (by Proposition 6.4 (2)).

(2) For all $z \in G$ we have

$$(r_{f,n})^n(z) = \exp(\frac{1}{n}L_f(z))^n \overset{\exp(w)^n = \exp(nw)}{=} \exp(L_f(z)) = f(z).$$

and

$$r'_{f,n}(z) = \frac{1}{n}\frac{f'(z)}{f(z)}r_{f,n}(z).$$

7 Properties of holomorphic functions II

(A) Biholomorphic maps and local description of holomorphic functions

Definition 7.1. Let $U, U' \subseteq \mathbb{C}$ be open.

(1) A bijective map $f\colon U \to U'$ is called *biholomorphic* or *conformal* if f and f^{-1} are holomorphic.

(2) A holomorphic function $f\colon U \to \mathbb{C}$ is called *locally biholomorphic* if for all $z \in U$ there exist open neighborhoods $z \in W \subseteq U$ and $f(z) \in W' \subseteq \mathbb{C}$ such that $f_{|W}\colon W \to W'$ is biholomorphic.

Remark 7.2. Let $U, U' \subseteq \mathbb{C}$ be open.

(1) A biholomorphic map $f\colon U \to U'$ is a homeomorphism.

(2) The Inverse Function Theorem (Theorem 1.9) shows that a holomorphic function $f\colon U \to \mathbb{C}$ is locally biholomorphic if and only if $f'(z) \neq 0$ for all $z \in U$.

(3) exp is locally biholomorphic on $\mathbb{C}$ (because $\exp'(z) = \exp(z) \neq 0$ for all $z \in \mathbb{C}$) but not biholomorphic.

Theorem 7.3. *Let $U \subseteq \mathbb{C}$ be open, $f\colon U \to \mathbb{C}$ holomorphic. Let $z_0 \in U$ and let*

$$f(z) = a_0 + \sum_{n=m}^{\infty} a_n(z - z_0)^n, \qquad \text{with } m \in \mathbb{N} \text{ and } a_m \neq 0$$

Then there exist open neighborhoods $0 \in W \subseteq \mathbb{C}$ and $z_0 \in W' \subseteq U$ and a biholomorphic map $\varphi\colon W \to W'$ such that $f(\varphi(w)) = a_0 + w^m$ for all $w \in W$.

In other words: Locally, after a change of charts $w \mapsto \varphi(w)$,f is of the form $f(w) = a_0 + w^m$.

Proof. Replacing f by $f - a_0$ and z by $z + z_0$ we may assume that $a_0 = 0$ and $z_0 = 0$. We are looking for open neighborhoods W and W' of 0 and a biholomorphic $\Phi\colon W' \to W$ such that $f(z) = \Phi(z)^m$ for $z \in W$ (then take $\varphi := \Phi^{-1}$). We have

$$f(z) = a_m z^m g(z),$$

where $g\colon U \to \mathbb{C}$ is holomorphic with $g(0) = 1$ (Lemma 5.20 (1)). Hence we can find $0 \in V' \subseteq U$ open such that $g(z) \neq 0$ for all $z \in V'$ and such that V' is an open disc and in particular simply connected. Then by Remark 6.9 there exists a holomorphic function $\tilde{g}\colon V' \to \mathbb{C}$ such that $\tilde{g}^m = g$. In particular $\tilde{g}(0) \neq 0$. Moreover let $a \in \mathbb{C}^\times$ with $a^m = a_m$. Then

$$f(z) = (\Phi(z))^m, \qquad \text{with } \Phi\colon V' \to \mathbb{C},\ \Phi(z) = az\tilde{g}(z).$$

Moreover $\Phi'(0) = a\tilde{g}(0) \neq 0$. Hence by Theorem 1.9 there exist $0 \in W' \subseteq V'$ open and $0 = \Phi(0) \in W \subseteq \mathbb{C}$ open such that $\Phi\colon W' \to W$ is biholomorphic. Set $\varphi := \Phi^{-1}$. □

Corollary 7.4. *Let $U \subseteq \mathbb{C}$ be open, $f\colon U \to \mathbb{C}$ holomorphic. Assume that f is injective. Then $V := f(U)$ is open in $\mathbb{C}$ and $f\colon U \to V$ is biholomorphic (in particular: $f'(z) \neq 0$ for all $z \in U$).*

Proof. As $f\colon U \to V$ is bijective, there exists an inverse map $g\colon V \to U$. It suffices to show that $f'(z_0) \neq 0$ for all $z_0 \in U$ ($\Rightarrow f$ locally biholomorphic $\Rightarrow V$ open and g holomorphic). Write $f(\varphi(w)) = a_0 + w^m$ as in Theorem 7.3. A look at the power series expansion $f(z) = \sum_{n \geq 0} a_n(z - z_0)^n$ shows that $f'(z_0) \neq 0$ if and only if $a_1 \neq 0$, i.e., if and only if $m = 1$. As φ is injective, $w \mapsto a_0 + w^m$ is injective on a neighborhood of 0. Hence $m = 1$ and therefore $f'(z_0) \neq 0$. □

Example 7.5. Let

$$\log\colon G := \{\, re^{i\varphi} \ ;\ r \in \mathbb{R}^0, -\pi < \varphi < \pi \,\} \to \mathbb{C}, \qquad re^{i\varphi} \mapsto \log(r) + i\varphi$$

be the principal branch of logarithm on G (Example 6.7.1). Then log is injective (because $\exp \circ \log = \mathrm{id}_G$) and hence log is a biholomorphic map from G onto the vertical strip $\{\, z \in \mathbb{C} \ ;\ -\pi < \mathrm{Im}(z) < \pi \,\}$ by Corollary 7.4.

(B) Open Mapping Theorem

Definition 7.6.

(1) Let X and Y be topological spaces. A map $f\colon X \to Y$ is called *open* if $f(U)$ is open in Y for every open subset U of X.

(2) Let X be a topological space. A *basis of of* X[5] is a set $\mathcal{B}$ of open subsets of X such that every open subset of X is a union of subsets in $\mathcal{B}$.

Remark 7.7.

(1) Let (X, d) be a metric space and let $X = \bigcup_{i \in I} U_i$ with $U_i \subseteq X$ open. Then

$$\mathcal{B} := \{ B_r(x_0) \ ; \ x_0 \in X, r \in \mathbb{R}^{>0}, \exists\, i \in I\colon B_r(x_0) \subseteq U_i \}$$

is a basis of the topological space X.

(2) Let $f\colon X \to Y$ be a map of topological spaces X and Y, let $\mathcal{B}$ be a basis of X. If $f(V)$ is open in Y for all $V \in \mathcal{B}$, then f is open.

Proof. Let $U \subseteq X$ be open. Then $U = \bigcup_{i \in I} U_i$ with $U_i \in \mathcal{B}$ and hence $f(U) = \bigcup_{i \in I} f(U_i)$ is open in Y. $\square$

(3) The composition of two open maps is again open.

(4) Every homeomorphism is open.

(5) Every locally biholomorphic map is open.

Theorem 7.8 (Open mapping theorem). *Let $G \subseteq \mathbb{C}$ be a domain and let $f\colon G \to \mathbb{C}$ be holomorphic and not constant. Then f is open.*

Proof. By Remark 7.7 (2) it suffices to check that for all $z_0 \in G$ there exists $z_0 \in U \subseteq G$ open such that $f(B_r(z_0))$ is open for all $r \in \mathbb{R}^{>0}$ with $B_r(z_0) \subseteq U$. Note that the Identity theorem implies: f not constant $\Rightarrow f_{|U}$ not constant for $\emptyset \neq U \subseteq G$ open. Thus by Theorem 7.3 we may assume that f is of the form $w \mapsto z_0 + w^m$ for some $m \in \mathbb{N}$. But this map clearly sends open discs to open discs. $\square$

(C) Maximum modulus principle

Recall: A subspace $I \subseteq \mathbb{R}^{\geq 0}$ is open and connected if and only if I is of the form $I = (a, b)$ for real numbers $0 < a < b$ or of the from $I = [0, b)$ for $b \in \mathbb{R}^{>0}$.

Example 7.9.

(1) Let X and Y be topological spaces and endow $X \times Y$ with the product topology. Then the projections $X \times Y \to X$, $(x, y) \mapsto x$ and $X \times Y \to Y$, $(x, y) \mapsto y$ are open.

(2) Let $(V, \|\cdot\|)$ be a normed $\mathbb{R}$-vector space. Then $\|\cdot\|\colon V \to \mathbb{R}^{\geq 0}$ is open.

Proof. Remark 7.7 (2) $\Rightarrow$ It suffices to show that the image of $B_r(v)$ under $\|\cdot\|$ is open in $\mathbb{R}^{\geq 0}$ for all $v \in V$ and $r \in \mathbb{R}^{>0}$. But this image is $(\|v\| - r, \|v\| + r) \cap \mathbb{R}^{\geq 0}$. $\square$

(3) In particular: The maps $\mathbb{C} \to \mathbb{R}$, $z \mapsto \mathrm{Re}(z)$ and $z \mapsto \mathrm{Im}(z)$ are open. The map $\mathbb{C} \to \mathbb{R}^{\geq 0}$, $z \mapsto |z|$ is open.

[5] This notion has nothing to do with the notion of a basis of a vector space.

Theorem 7.10 (Maximum/Minimum modulus principle). *Let $G \subseteq \mathbb{C}$ be a domain, $f\colon G \to \mathbb{C}$ a holomorphic function. Assume that there exists $z_0 \in G$ and $z_0 \in W \subseteq G$ open such that* one *of the following conditions hold.*
(a) $|f(z)| \le |f(z_0)|$ *for all $z \in W$ (i.e. $|f|$ has a local maximum in z_0).*
(b) $|f(z)| \ge |f(z_0)|$ *for all $z \in W$ and $f(z_0) \neq 0$.*
Then f is constant.

Proof. Assume f is non-constant. Then f is open (Theorem 7.8). As $|\cdot|\colon \mathbb{C} \to \mathbb{R}^{\ge 0}$ is also open (Example 7.9 (2)), $|f|(W)$ is open in $\mathbb{R}^{\ge 0}$. But under assumptions (a) or (b) the point $|f(z_0)| \in |f|(W)$ is not an inner point. Contradiction. □

Note: Under assumption (b) one has to exclude $|f(z_0)| = 0$ because sets of the form $[0, b)$ are open in $\mathbb{R}^{\ge 0}$.

Corollary 7.11. *Let $G \subseteq \mathbb{C}$ be a bounded domain, let $\bar{G}$ be its closure. Let $f\colon \bar{G} \to \mathbb{C}$ be a continuous function such that $f_{|G}$ is holomorphic. Then*

$$\sup_{z \in \bar{G}} |f(z)| = \max_{z \in \partial G} |f(z)|,$$

i.e., $|f|$ attains its maximum on ∂G.

Proof. If f is constant, the assertion is trivial. Hence assume that f is non-constant. As G is bounded, $\bar{G}$ and ∂G are compact. Therefore $|f|$ attains its maximum on $\bar{G}$. But this cannot be in G if f is non-constant (Theorem 7.10). □

Remark 7.12. As $\operatorname{Re}\colon \mathbb{C} \to \mathbb{R}$ is open (Remark 7.9), the same argument as in the proof of Theorem 7.10 shows: G domain, $f\colon G \to \mathbb{C}$ holomorphic. Assume there exists $z_0 \in G$ such that $\operatorname{Re}(f(z_0)) \ge \operatorname{Re}(f(z))$ for all $z \in G$ or that $\operatorname{Re}(f(z_0)) \le \operatorname{Re}(f(z))$ for all $z \in G$. Then f is constant.
Similar for the imaginary part.

8 Homology and the winding number

(A) Digression: Covering Spaces

Notation: In this section, X and $\tilde{X}$ always denote Hausdorff topological spaces.

Definition 8.1.
(1) A continuous map $p\colon \tilde{X} \to X$ is called a *covering* if for all $x \in X$ there exists $x \in U \subseteq X$ open such that

(8.1.1) $$p^{-1}(U) = \bigsqcup_{i \in I} \tilde{U}_i \quad \textit{disjoint} \text{ union, } I \neq \emptyset \text{ some index set}$$

for open sets $\tilde{U}_i \subseteq X$ such that $p_{|\tilde{U}_i}\colon \tilde{U}_i \to U$ is a homeomorphism for all $i \in I$.
(2) A covering $p\colon \tilde{X} \to X$ is called a *universal covering* if $\tilde{X}$ is simply connected.

Remark 8.2.

(1) A covering is always a surjective map: For every U as in (8.1.1) and for every $y \in U$ we have a bijection $p^{-1}(\{y\}) \leftrightarrow I$.
(2) The map $f\colon \mathbb{C} \to \mathbb{C}$, $z \mapsto z^2$ is surjective, but there exists no open neighborhood U of 0 such that $\#f^{-1}(\{z_0\}) = \#f^{-1}(\{0\}) = 1$ for all $z_0 \in U$.

Example 8.3.
(1) The function $\mathbb{R} \to S^1 = \{\, z \in \mathbb{C} \ ; \ |z| = 1 \,\}$, $x \mapsto e^{2\pi i x}$ is a universal covering.
(2) The function $\exp\colon \mathbb{C} \to \mathbb{C}^\times$ is a universal covering.

We will only prove (and use) (2). (1) is left as an exercise.

Complex analytic proof. As $\mathbb{C}$ is simply connected, we only have to show that exp is a covering. For $y \in \mathbb{C}^\times$ let $V \subseteq \mathbb{C}^\times$ be simply connected domain with $y \in V$. Let $L\colon V \to \mathbb{C}$ be a branch of the logarithm (i.e. $\exp(L(z)) = z$ for all $z \in V$). Then an arbitrary branch of the logarithm on V is given by $L_k := L + 2\pi i k$ for $k \in \mathbb{Z}$ (Corollary 6.5). Set $U_k := L_k(V)$. Then U_k is open in $\mathbb{C}$ by the Open mapping Theorem 7.8. Then $L_k\colon V \to U_k$ is bijective and holomorphic with inverse map $\exp_{|U_k}\colon U_k \to V$ (Corollary 7.4). Moreover

$$\exp^{-1}(V) = \bigsqcup_{k \in \mathbb{Z}} U_k$$

is a disjoint union of open sets (if there exists $z \in U_k \cap U_l$ then $z = L_k(\exp(z)) = L_l(\exp(z)) = L_k(\exp(z)) + 2\pi i(l-k)$ hence $l = k$). □

Proposition 8.4. *Let $p\colon \tilde{X} \to X$ be a covering, let $\gamma\colon [a,b] \to X$ be a path, and let $\tilde{x} \in \tilde{X}$ such that $p(\tilde{x}) = \gamma(a)$ (such $\tilde{x}$ always exists because p is surjective). Then there exists a unique path $\tilde{\gamma}\colon [a,b] \to \tilde{X}$ such that*

$$(*) \qquad \tilde{\gamma}(a) = \tilde{x} \qquad \text{and} \qquad p \circ \tilde{\gamma} = \gamma.$$

Note, that even if γ is a loop, then $\tilde{\gamma}$ is not necessarily a loop.

Proof. Uniqueness. Let $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ be two paths satisfying (*). Then $Z := \{\, t \in [a,b] \ ; \ \tilde{\gamma}_1(t) = \tilde{\gamma}_2(t) \,\}$ is non-empty. It is closed because $\tilde{X}$ is Hausdorff. Moreover for $t \in Z$ choose an open neighborhood U of $p(\tilde{\gamma}_i(t))$ as in (8.1.1). By continuity, $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ must map a neighborhood $t \in W \subseteq [a,b]$ into the same $\tilde{U}_i$. Hence $p \circ \tilde{\gamma}_1 = p \circ \tilde{\gamma}_2$ shows $\tilde{\gamma}_{1|W} = \tilde{\gamma}_{2|W}$. This shows that Z is also open. Hence $Z = [a,b]$ because $[a,b]$ is connected.
Existence. The existence of $\tilde{\gamma}$ is clear if $\gamma([a,b]) \subseteq U$ with U as in (8.1.1).
In general the compactness of $[a,b]$ shows that there exist $a = a_0 < a_1 < \cdots < a_m = b$ such that for all $j = 1, \dots, m$ one has $\gamma([a_{j-1}, a_j]) \subseteq U$ for some U as in (8.1.1). Hence we can lift $\gamma_{|[a_{j-1},a_j]}$ successively. □

(B) The winding number

Proposition 8.5. *Let $z_0 \in \mathbb{C}$ and let γ be a loop in $\mathbb{C} \setminus \{z_0\}$. Then there exists $k \in \mathbb{Z}$ such that γ is loop homotopic in $\mathbb{C}^\times$ to the loop*

$$\beta_k \colon [0,1] \to \mathbb{C}^\times, \qquad t \mapsto z_0 + \exp(2\pi i k t).$$

Proof. We may assume that $z_0 = 0$. By Remark 3.4 (7) it suffices to show that every loop γ with startpoint 1 is homotopic to some β_k. As exp is a covering (Example 8.3), there exists a unique lift $\tilde{\gamma} \colon [0,1] \to \mathbb{C}$ such that $\tilde{\gamma}(0) = 0$ (Proposition 8.4). Then $\exp(\tilde{\gamma}(1)) = 1$ and hence there exists $k \in \mathbb{Z}$ such that $\tilde{\gamma}(1) = 2\pi i k$. As $\mathbb{C}$ is simply connected, there exists a homotopy $\tilde{H}_k$ of $\tilde{\gamma}$ and the line segment σ_k from 0 to $2\pi i k$ (i.e., $\sigma_k \colon [0,1] \to \mathbb{C}$, $\sigma_k(t) = 2\pi i k t$). But then $\exp \circ \tilde{H}$ is a homotopy of γ and $\exp \circ \sigma_k = \beta_k$. □

In fact, the integer k then can be computed (by homotopy invariance of the integral) by

$$\frac{1}{2\pi i} \int_\gamma \frac{1}{z - z_0}\, dz = \frac{1}{2\pi i} \int_{\beta_k} \frac{1}{z - z_0}\, dz \overset{4.6}{=} k.$$

This leads us to the following definition.

Definition 8.6. Let $\gamma \colon [a,b] \to \mathbb{C}$ be a path and let $u \in \mathbb{C} \setminus \{\gamma\}$. Then

$$W(\gamma; u) := \frac{1}{2\pi i} \int_\gamma \frac{1}{z - u}\, dz$$

is called the *winding number* (German: *Umlaufzahl* or *Windungszahl*) or the *index of γ with respect to w*.

Note: $z \mapsto 1/(z - u)$ is holomorphic on the open neighborhood $\mathbb{C} \setminus \{u\}$ of $\{\gamma\}$, thus $\int_\gamma \frac{1}{z-u}\, dz$ is defined for every path γ (Remark 4.7).

Proposition 8.7. *Let $\gamma \colon [a,b] \to \mathbb{C}$ be a loop.*
(1) *$W(\gamma; u)$ is an integer for all $u \in \mathbb{C} \setminus \{\gamma\}$.*
(2) *The function*

$$\mathbb{C} \setminus \{\gamma\} \to \mathbb{Z}, \qquad u \mapsto W(\gamma; u)$$

is locally constant (and hence constant on the path components of $\mathbb{C} \setminus \{\gamma\}$).
(3) *Let $G \subseteq \mathbb{C} \setminus \{\gamma\}$ be an*[6] *unbounded path component. Then $W(\gamma; u) = 0$ for all $u \in G$.*
(4) *Let $u \in \mathbb{C}$ and let γ and δ be loops that are loop homotopic in $\mathbb{C} \setminus \{u\}$. Then $W(\gamma; u) = W(\delta; u)$.*

[6] In fact, using that $\{\gamma\}$ is compact, it is easy to see that there is only one unbounded path component.

Proof. (1). Proposition 8.5.
(2). The function is continuous (even holomorphic) by Lemma 5.7 and $\mathbb{Z}$-valued by (1). Hence it is locally constant.
(3). As G is not bounded, there exists for all $\varepsilon > 0$ an $u \in G$ such that $\sup\{\, 1/(z - u)\ ;\ z \in \{\gamma\}\,\} \leq \varepsilon$. Hence $W(\gamma; u) \leq \varepsilon L(\gamma)$. Hence $W(\gamma; u) = 0$ for all $u \in G$ by (2).
(4). Homotopy invariance of the path integral of holomorphic functions (Theorem 4.11). □

Example 8.8. Let $z_0 \in \mathbb{C}$, $r \in \mathbb{R}^{>0}$, and $\gamma = \partial B_r(z_0)$. Then we have for $u \in \mathbb{C} \setminus \{\gamma\}$:

$$W(\gamma; u) = \begin{cases} 1, & u \in B_r(z_0); \\ 0, & u \notin \overline{B_r(z_0)}. \end{cases}$$

(C) First Homology group

Notation: In this subsection let X be a topological space.

Definition 8.9. Let S be a set. The *free abelian group generated by S* is the abelian group

$$\mathbb{Z}^{(S)} := \{\, n\colon S \to \mathbb{Z}\ ;\ n \text{ map } s \mapsto n_s \text{ with } n_s = 0 \text{ for all but finitely many } s \in S\}.$$

Elements $n \in \mathbb{Z}^{(S)}$ are usual written as $\sum_{s\in S} n_s s$ (a "formal linear combination"). In practice, we skip the summands with $n_s = 0$ and write also $n_1 s_1 + \cdots + n_r s_r$ with $n_i \in \mathbb{Z}$, $s_i \in S$, $r \in \mathbb{N}_0$.
Using the notation as formal linear combinations, the addition in $\mathbb{Z}^{(S)}$ is given by

$$\sum_{s\in S} n_s s + \sum_{s\in S} m_s s = \sum_{s\in S} (n_s + m_s) s.$$

$\mathbb{Z}^{(S)}$ is a free $\mathbb{Z}$-module with basis $s \in S$.
Instead of $\mathbb{Z}$ we can also take a field K and obtain a vector space $K^{(S)}$. (In fact, we may take any ring R and obtain the *free left R-module generated by S*.)

Definition 8.10. Let $C_1(X, \mathbb{Z})$ be the free abelian group generated by $S := \mathcal{C}([0,1], X)$, the set of paths in X. An element $\Gamma \in C_1(X, \mathbb{Z})$ is called a *1-chain in X*. It is written as $\Gamma = \sum_\gamma n_\gamma \gamma$ or simply as $n_1\gamma_1 + \cdots + n_r\gamma_r$ with $\gamma_i\colon [0,1] \to X$ a path and $n_i \in \mathbb{Z}$.
We consider paths $\gamma\colon [0,1] \to X$ as the element $1 \cdot \gamma \in C_1(X, \mathbb{Z})$.
For $\Gamma = n_1\gamma_1 + \cdots + n_r\gamma_r \in C_1(X, \mathbb{Z})$ we define

$$\{\Gamma\} := \bigcup_{\substack{1\leq i\leq r \\ n_i \neq 0}} \{\gamma_i\}.$$

Definition 8.11. (1) A *0-chain in X* is an element of $C_0(X, \mathbb{Z}) := \mathbb{Z}^{(X)}$, i.e., 0-chains are formal linear combinations $n_1 x_1 + \ldots n_r x_r$ with $n_i \in \mathbb{Z}$ and $x_i \in X$.

(2) Let $\Delta := \{ (x,y) \in \mathbb{R}^2 \ ; \ 0 \leq y \leq x \leq 1 \}$ be the triangle in $\mathbb{R}^2$ with vertices $(0,0)$, $(1,0)$, and $(1,1)$.
A *2-chain in* X is an element of $C_2(X,\mathbb{Z}) := \mathbb{Z}^{(\mathcal{C}(\Delta,X))}$, i.e., 2-chains are formal linear combinations $n_1\delta_1 + \dots n_r\delta_r$ with $n_i \in \mathbb{Z}$ and $\delta_i \colon \Delta \to X$ continuous.

Definition 8.12. (1) Let $\gamma\colon [0,1] \to X$ be a path. Define $\partial_1(\gamma) := \gamma(1) + (-1)\gamma(0) \in C_0(X,\mathbb{Z})$. More generally, define a map

$$\partial_1 \colon C_1(X,\mathbb{Z}) \to C_0(X,\mathbb{Z}), \qquad \sum_{i=1}^{r} n_i\gamma_i \mapsto \sum_{i=1}^{r} n_i(\gamma_i(1) - \gamma_i(0)).$$

This is a group homomorphism.

(2) Let $\Delta = \{ (x,y) \in \mathbb{R}^2 \ ; \ 0 \leq y \leq x \leq 1 \}$ as above. We describe its boundary with 3 paths:

$$\begin{aligned}
(\partial\Delta)_1 &\colon [0,1] \to \mathbb{R}^2, & t &\mapsto (t,0),\\
(\partial\Delta)_2 &\colon [0,1] \to \mathbb{R}^2, & t &\mapsto (1,t),\\
(\partial\Delta)_3 &\colon [0,1] \to \mathbb{R}^2, & t &\mapsto (1-t,1-t).
\end{aligned}$$

Let $\delta\colon \Delta \to X$ be continuous. Define

$$\partial_2(\delta) := \sum_{k=1}^{3} \delta \circ (\partial\Delta)_k \in C_1(X,\mathbb{Z}).$$

More generally, define

$$\partial_2 \colon C_2(X,\mathbb{Z}) \to C_1(X,\mathbb{Z}), \qquad \sum_{i=1}^{r} n_i\delta_i \mapsto \sum_{i=1}^{r} n_i\partial_2(\delta_i).$$

This is a group homomorphism.

The homomorphisms ∂_1 and ∂_2 are called *boundary maps*. We also define $\partial_0 \colon C_0(X,\mathbb{Z}) \to 0$ the zero homomorphism and therefore have group homomorphisms

$$C_2(X,\mathbb{Z}) \xrightarrow{\partial_2} C_1(X,\mathbb{Z}) \xrightarrow{\partial_1} C_0(X,\mathbb{Z}) \xrightarrow{\partial_0} 0.$$

For $\delta\colon \Delta \to X$ one has $\partial_1(\partial_2(\delta)) = 0$ and hence

$$\partial_1 \circ \partial_2 = 0. \tag{8.12.1}$$

Definition and Remark 8.13. Define for $i = 0,1$ subgroups of $C_i(X,\mathbb{Z})$:

$$Z_i(X,\mathbb{Z}) := \ker(\partial_i), \qquad B_i(X,\mathbb{Z}) := \operatorname{im}(\partial_{i+1}).$$

Elements of $Z_i(X,\mathbb{Z})$ are called *i-cycles*. Elements of $B_i(X,\mathbb{Z})$ are called *i-boundaries*.
We have:

(1) $Z_0(X,\mathbb{Z}) = C_0(X,\mathbb{Z})$.

(2) $Z_1(X,\mathbb{C})$ consists of those 1-chains $\sum_{i=1}^r n_i\gamma_i$ such that every point $x \in X$ one has

$$\sum_{\substack{1\le i\le r\\ x=\gamma_i(0)}} n_i = \sum_{\substack{1\le i\le r\\ x=\gamma_i(1)}} n_i,$$

i.e., such that every point is as often startpoint as it is an endpoint.

(3) As $\partial_1 \circ \partial_2 = 0$,

$$B_1(X,\mathbb{Z}) \subseteq Z_1(X,\mathbb{Z}) \subseteq C_1(X,\mathbb{Z}).$$

For $i = 0,1$ we call $H_i(X,\mathbb{Z}) := Z_i(X,\mathbb{Z})/B_i(X,\mathbb{Z})$ the *i-th singular homology group of X*. Two i-chains $\Gamma, \Gamma' \in C_i(X,\mathbb{Z})$ are called *homologous in* X, if $\Gamma - \Gamma' \in B_i(X,\mathbb{Z})$. An i-chain Γ is called *null-homologous in* U, if $\Gamma \in B_i(X,\mathbb{Z})$.

Remark 8.14. The map $X \to \pi_0(X)$ that sends a point $x \in X$ to the path component of X containing x yields an isomorphism $H_0(X,\mathbb{Z}) \xrightarrow{\sim} \mathbb{Z}^{(\pi_0(X))}$ of groups. In other words: Two points $x, x' \in X$ are homologous if and only x and x' lie in the same path-component of X.

Proof. Exercise $\square$

(D) Homology, winding numbers, and homotopy

Definition 8.15. Let $X = V$, $W = \mathbb{R}^m$ be finite-dimensional $\mathbb{R}$-vector spaces.

(1) A 1-chain $\Gamma = n_1\gamma_1 + \dots + n_r\gamma_r$ in V is called *piecewise $\mathcal{C}^1$* if every path γ_i with $n_i \neq 0$ is piecewise $\mathcal{C}^1$.

(2) Let $\Gamma = n_1\gamma_1 + \dots n_r\gamma_r$ be a 1-chain in V, let $\omega\colon \{\Gamma\} \to \mathrm{Hom}_{\mathbb{R}}(V,W)$ be a continuous W-valued 1-form. Assume that Γ is piecewise $\mathcal{C}^1$ or that there exists an open neighborhood U of $\{\Gamma\}$ and a closed W-valued 1-form $\tilde\omega$ on U such that $\tilde\omega_{|\{\Gamma\}} = \omega$. Then define

$$\int_\Gamma \omega := \sum_{i=1}^r n_i \int_{\gamma_i} \omega.$$

(3) Let $V = W = \mathbb{C}$, $u \in \mathbb{C}$ and let Γ be a 1-chain in $\mathbb{C} \setminus \{u\}$ (i.e. $\{\Gamma\} \subseteq \mathbb{C} \setminus \{u\}$). Then define the *winding number*

$$W(\Gamma; u) := \frac{1}{2\pi i}\int_\Gamma \frac{1}{z-u}\,dz.$$

Proposition 8.16. *Let $U \subseteq \mathbb{C}$ be open, and let $\Gamma, \Gamma' \in C_1(U,\mathbb{Z})$ be two 1-chains in U. Then Γ and Γ' are homologous in U if and only if*

$$(*)\qquad \begin{aligned} &\partial_1(\Gamma) = \partial_1(\Gamma') \\ \textit{and}\quad &W(\Gamma;u) = W(\Gamma';u) \qquad \forall\, u \in \mathbb{C}\setminus U. \end{aligned}$$

In particular we have for $\Gamma \in Z_1(U,\mathbb{Z})$:

$$\Gamma \textit{ null-homologous in } U \Leftrightarrow W(\Gamma;u) = 0 \quad \forall\, u \in \mathbb{C}\setminus U.$$

The hypothesis "$\partial_1(\Gamma) = \partial_1(\Gamma')$" is in particular satisfied if $\Gamma, \Gamma' \in Z_1(U, \mathbb{Z})$.
In the sequel we will take (*) as a definition of "homologous" and we will not use Proposition 8.16.

Proof. W. Fulton: *Algebraic Topology, A first course*, Springer (1995), Theorem 6.11 + Exercise 6.13 □

Proposition 8.17. *Let X be a topological space and let $\gamma, \gamma' \colon [0,1] \to U$ be paths in X. If γ and γ' are homotopic, then γ and γ' are homologous.*

There exist open subsets U of $\mathbb{C}$ and loops in U that are null-homologous in U but not null-homotopic in U.
We will prove the proposition only if $X = U$ is open in $\mathbb{C}$. For a proof that works in the general case, see: W. Fulton: *Algebraic Topology, A first course*, Springer (1995), Lemma 6.4.

Proof. If γ and γ' are homotopic in U, then

$$\partial_1(\gamma) = \gamma(1) - \gamma(0) = \gamma'(1) - \gamma'(0) = \partial_1(\gamma') \in C_0(U, \mathbb{Z}).$$

Moreover, Cauchy's theorem (Theorem 4.11) shows that $W(\gamma; u) = W(\gamma'; u)$ for $u \in \mathbb{C} \setminus U$. Therefore Proposition 8.16 shows that γ and γ' are homologous. □

Remark 8.18. Let X be a topological space. Proposition 8.17 shows that we obtain for $x_0 \in X$ a well defined map

$$h := h_{X,x_0} \colon \pi_1(X, x_0) \to H_1(X, \mathbb{Z}), \qquad [\gamma] \mapsto \gamma + B_1(U, \mathbb{Z}) \tag{8.18.1}$$

We claim that this is a group homomorphism. We show the claim again only if $X = U$ is open in $\mathbb{C}$. For $\gamma, \delta \in \pi_1(U, x_0)$ the loop $\gamma \cdot \delta$ and the 1-cycle $\gamma + \delta$ have the same winding numbers and hence they are homologous. This shows that h_{U,x_0} is a group homomorphism.

Proposition 8.19. *Let X be a path-connected topological space, let $x \in X$. Consider the group homomorphism $h_{X,x} \colon \pi_1(X, x) \to H_1(X, \mathbb{Z})$ (8.18.1).*
(1) $h_{X,x}$ is surjective
(2) Its kernel of $h_{X,x}$ is the derived group $\pi_1(X, x)^{\mathrm{der}}$ of $\pi_1(X, x)$, i.e. the subgroup generated by $\gamma\delta\gamma^{-1}\delta^{-1}$ for $\gamma, \delta \in \pi_1(X, x)$.
Therefore $h_{X,x}$ induces an isomorphism of abelian groups

$$\pi_1(X, x)^{\mathrm{ab}} := \pi_1(X, x)/\pi_1(X, x)^{\mathrm{der}} \xrightarrow{\sim} H_1(X, \mathbb{Z}).$$

Proof. We will only prove (and use in the sequel) the surjectivity of $h_{X,x}$ (for a proof of Assertion (2) see: W. Fulton: *Algebraic Topology, A first course*, Springer (1995), Theorem 12.22).
Let $\Gamma = \sum_{i=1}^{r} n_i \gamma_i \in Z_1(X, \mathbb{Z})$, where $n_i \in \mathbb{Z}$ and γ_i a path with startpoint $p_i \in X$ and endpoint $q_i \in X$. For each point $c \in X$ that occurs as an endpoint or as a startpoint of any γ_i, choose a path τ_c from x to c (this is possible because X is path connected). Set

$$\tilde{\gamma}_i := \tau_{p_i} \cdot \gamma_i \cdot \tau_{q_i}^{-}.$$

This is a loop with startpoint x. Let $[\tilde{\gamma}_i] \in \pi_1(X,x)$ be its homotopy class and let

$$\gamma := [\tilde{\gamma}_1]^{n_1} \cdots [\tilde{\gamma}_1]^{n_r} \in \pi_1(X,x).$$

Then $h_{X,x}$ sends γ to the following element in $H_1(X,\mathbb{Z})$:

$$\sum_{i=1}^{r} n_i \tilde{\gamma}_i = \sum_{i=1}^{r} n_i(-\tau_{q_i} + \gamma_i + \tau_{p_i}) \overset{\Gamma \in Z_1}{=} \sum_{i=1}^{r} n_i \gamma_i = \Gamma \in H_1(X,\mathbb{Z})$$

This shows the surjectivity of $h_{X,x}$. □

Corollary 8.20. *Let X be a simply connected topological space. Then $H_1(X,\mathbb{Z}) = 0$.*

Proof. Choose $x \in X$. X simply connected $\Leftrightarrow \pi_1(X,x) = 0$, hence $H_1(X,\mathbb{Z}) = 0$ by Proposition 8.19. □

Corollary 8.21. *Let $G \subseteq \mathbb{C}$ be a domain. Then every 1-cycle in $Z_1(G,\mathbb{Z})$ is homologous to a piecewise $\mathcal{C}^1$-loop in G.*

Proof. By Proposition 8.19 every 1-cycle in G is homologous to a loop in G. Every loop γ in G is homotopic in G to a piecewise $\mathcal{C}^1$-loop γ' (Remark 3.15). In particular γ is homologous to γ'. □

Proposition 8.22. *Let $\Gamma \in Z_1(\mathbb{C},\mathbb{Z})$.*
(1) *$W(\Gamma;u) \in \mathbb{Z}$.*
(2) *The map $\mathbb{C} \setminus \{\Gamma\} \to \mathbb{Z}$, $u \mapsto W(\Gamma;u)$ is locally constant.*
(3) *Let $G \subseteq \mathbb{C}\setminus\{\Gamma\}$ be an unbounded path component. Then $W(\Gamma;u) = 0$ for all $u \in G$.*

Proof. (1),(3). Applying Corollary 8.21 for $G = \mathbb{C} \setminus \{u\}$ we may assume that Γ is a loop. Hence this follows from Proposition 8.7.
(2). It suffices to show that for all $u \in \mathbb{C}\setminus\{\Gamma\}$ there exists $r > 0$ with $\overline{B_r(u)} \subset \mathbb{C}\setminus\{\Gamma\}$ such that $B_r(u) \to \mathbb{Z}$, $z \mapsto W(\Gamma;z)$ is constant. Using Corollary 8.21 for $G = \mathbb{C}\setminus\overline{B_r(u)}$ we may assume that Γ is a loop. Then we are again done by Proposition 8.7. □

Remark 8.23. Let $U \subseteq \mathbb{C}$ be open and $\Gamma \in Z_1(U,\mathbb{Z})$. Then $\{\Gamma\}$ is compact and hence $\{\Gamma\} \subseteq B_R(0)$ for some $R \in \mathbb{R}^{>0}$. Then Remark 8.22 (3) implies

$$\overline{\{\, u \in \mathbb{C} \setminus \{\Gamma\} \ ; \ W(\Gamma;u) \neq 0 \,\}} \subseteq \overline{B_R(0)}.$$

In particular, it is compact.

Example 8.24. Let $u \in \mathbb{C}$ and set $G := \mathbb{C} \setminus \{u\}$.
(1) As $W(\Gamma;u) = 0$ for all $\Gamma \in B_1(G,\mathbb{Z})$ (Proposition 8.16), we obtain a group homomorphism

$$W(\cdot;u)\colon H_1(G,\mathbb{Z}) \to \mathbb{Z}, \qquad \Gamma \mapsto W(\Gamma;u).$$

This homomorphism is surjective because $n\partial B_1(u) \in Z_1(G,\mathbb{Z})$ has winding number n for all $n \in \mathbb{Z}$.

(2) Choose $z_0 \in G$. We obtain a surjective group homomorphisms

$$\pi_1(G, z_0) \xrightarrow{h} H_1(G, \mathbb{Z}) \overset{W(\cdot;u)}{\longrightarrow} \mathbb{Z}.$$

By Proposition 8.5 the composition $\pi_1(G, \mathbb{Z}) \to \mathbb{Z}$ is an isomorphism of groups. Hence h and $W(\cdot; u)$ are isomorphisms. Hence

$$\pi_1(G, z_0) \cong H_1(G, \mathbb{Z}) \cong \mathbb{Z}.$$

Example 8.25. Let $U \subseteq \mathbb{C}$ be open. Let $\Gamma \in B_1(U, \mathbb{Z})$ (e.g., if U is simply connected and $\Gamma \in Z_1(U, \mathbb{C})$). Let $z_1, \dots, z_m \in U \setminus \{\Gamma\}$ be distinct points and for $i = 1, \dots, m$ choose $r_i \in \mathbb{R}^{>0}$ such that $\overline{B_{r_i}(z_i)} \subseteq U$. Assume that $\overline{B_{r_i}(z_i)} \cap \overline{B_{r_j}(z_j)} = \emptyset$ for $i \neq j$. Then Γ is homologous in $U^* := U \setminus \{z_1, \dots, z_m\}$ to

$$\sum_{i=1}^{m} W(\Gamma; z_i) \partial B_{r_i}(z_i).$$

Proof. Set $n_i := W(\Gamma; z_i)$ and $\gamma_i = \partial B_{r_i}(z_i)$ for $i = 1, \dots, m$. We have to show that $W(\Gamma; u) = \sum_{i=1}^{m} n_i W(\gamma_i; u)$ for all $u \in \mathbb{C} \setminus U^* = (\mathbb{C} \setminus U) \cup \{z_1, \dots, z_m\}$.
For $u \in \mathbb{C} \setminus U$ we have $W(\Gamma; u) = 0$ by hypothesis and we have $W(\gamma_i; u) = 0$ for all i because u is outside every circle γ_i. Hence

$$W(\Gamma; u) = 0 = \sum_{n_i} W(\gamma_i; u).$$

If $u = z_k$ for some k, then $W(\gamma_i, z_k) = 1$ for $i = k$ and 0 for $i \neq k$. Hence

$$W(\Gamma; z_k) = n_k = \sum_{i=1}^{m} n_i W(\gamma_i; z_k). \qquad \square$$

(E) Cauchy's theorem, homology version

Theorem 8.26 (Cauchy integral formula, homology version). *Let $U \subseteq \mathbb{C}$ be open, let $\Gamma \in B_1(U, \mathbb{Z})$, and let $f \colon U \to \mathbb{C}$ be holomorphic. Let $z_0 \in U$ with $z_0 \notin \{\Gamma\}$. Then for all $n \in \mathbb{N}$ one has*

$$(*) \qquad W(\Gamma; z_0) f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_{\Gamma} \frac{f(z)}{(z - z_0)^{n+1}}\, dz.$$

Proof. (i). We may assume that $n = 0$: the general claim follows by differentiating under the integral sign (Proposition 4.10). Then the left hand side of (*) has the form $\frac{1}{2\pi i} \int_\Gamma \frac{f(z_0)}{z - z_0}\, dz$. Hence we have to prove:

$$h_0(z_0) := \int_{\Gamma} \frac{f(z) - f(z_0)}{z - z_0}\, dz = 0$$

for $z_0 \in U \setminus \{\Gamma\}$.

(ii). Define

$$g\colon U \times U \to \mathbb{C}, \qquad g(z,w) := \begin{cases} \frac{f(z)-f(w)}{z-w}, & z \neq w; \\ f'(z), & z = w. \end{cases}$$

Then g is continuous (Analysis 2). Moreover, $U \to \mathbb{C}$, $w \mapsto g(z,w)$ is continuous and holomorphic on $U \setminus \{z\}$. Thus it is holomorphic by Theorem 5.13.
(iii). Consider the continuous function

$$h_0\colon U \to \mathbb{C}, \qquad h_0(w) := \int_\Gamma g(z,w)\,dz.$$

It suffices to show that $h_0 = 0$.
We first claim that h_0 is holomorphic. This is a local question. Hence choose for all $z_0 \in U$ a disc $B_r(z_0) \subseteq U$ and let γ be a piecewise $\mathcal{C}^1$-loop in $B_r(z_0)$. Then

$$\int_\gamma h_0(w)\,dw = \int_\gamma \int_\Gamma g(z,w)\,dz\,dw \overset{\text{Fubini}}{=} \int_\Gamma \int_\gamma g(z,w)\,dw\,dz \overset{(ii)}{=} 0$$

because $w \mapsto g(z,w)$ is holomorphic. Hence h_0 is holomorphic on $B_r(z_0)$ by Corollary 5.12.
(iv). Extend h_0 to an entire function: Let $V := \{\, z \in \mathbb{C} \setminus \{\Gamma\} \;;\; W(\Gamma; z) = 0 \,\}$. As $\Gamma \in B_1(U, \mathbb{Z})$, we have $U \cup V = \mathbb{C}$. Define

$$h_1\colon V \to \mathbb{C}, \qquad h_1(w) := \int_\Gamma \frac{f(z)}{z-w}\,dz.$$

This is a holomorphic function by Lemma 5.7. For $w \in U \cap V$ we have

$$h_0(w) = \int_\Gamma \frac{f(z)}{z-w}\,dz - f(w)2\pi i W(\Gamma; w) = h_1(w).$$

Hence h_0 can be extended to an entire function $h\colon \mathbb{C} \to \mathbb{C}$ by setting

$$h(z) := \begin{cases} h_0(z), & \text{for } z \in U; \\ h_1(z), & \text{for } z \in V. \end{cases}$$

(v). Show $h = 0$: For all w in the unbounded path component of $\mathbb{C} \setminus \{\Gamma\}$ we have $W(\Gamma; w) = 0$ (Remark 8.22). Hence there exists $C \in \mathbb{R}^{\geq 0}$ depending only on Γ and $\|f\|_\Gamma$[7] such that we have for large $|w|$:

$$|h(w)| = |h_1(w)| \leq C \Big\| \frac{1}{z-w} \Big\|_\Gamma \xrightarrow{|w| \to \infty} 0.$$

Hence $h = 0$ by Liouville's theorem (Corollary 5.18). □

[7] Here $\|f\|_\Gamma := \sup\{\, |f(z)| \;;\; z \in \{\Gamma\} \,\}$.

Theorem 8.27 (Cauchy's theorem, homology version). *Let $U \subseteq \mathbb{C}$ be open and let $\Gamma \in Z_1(U, \mathbb{Z})$ be a 1-cycle in U. Then the following assertions are equivalent.*
(i) $\Gamma \in B_1(U, \mathbb{Z})$ *(i.e., Γ is nullhomologous in U).*
(ii) *For all holomorphic functions $f\colon U \to \mathbb{C}$ one has*

$$\int_\Gamma f\, dz = 0.$$

(iii) *For all $n \geq 1$ and all closed $\mathbb{R}^n$-valued 1-forms ω on U one has*

$$\int_\Gamma \omega = 0.$$

Proof. We will only show (und later use) the equivalence of (i) and (ii).
"(ii) $\Rightarrow$ (i)": For all $u \in \mathbb{C} \setminus U$ we have

$$W(\Gamma; u) = \int_\Gamma \frac{1}{z-u}\, dz \overset{(ii)}{=} 0.$$

"(i) $\Rightarrow$ (ii)": Let $f\colon U \to \mathbb{C}$ be holomorphic. Let $a \in U \setminus \{\Gamma\}$. Then $F\colon U \to \mathbb{C}$, $F(z) = (z-a)f(z)$ is holomorphic with $F(a) = 0$. Hence by Theorem 8.26 we have

$$0 = W(\Gamma; a)F(a) = \frac{1}{2\pi i}\int_\Gamma \frac{F(z)}{z-a}\, dz = \frac{1}{2\pi i}\int_\Gamma f(z)\, dz. \qquad \square$$

Corollary 8.28. *Let $U \subseteq \mathbb{C}$ be open, $f\colon U \to \mathbb{C}$ holomorphic, and let $\Gamma, \Gamma' \in C_1(U, \mathbb{Z})$ be 1-chains. If Γ and Γ' are homologous, then*

$$\int_\Gamma f(z)\, dz = \int_{\Gamma'} f(z)\, dz.$$

Remark 8.29 (Poincaré duality). Let $U \subseteq \mathbb{C}$ be open. Set

$$H^1_{\mathrm{DR}}(U, \mathbb{C}) := \{\mathbb{C}\text{-valued closed 1-forms on } U\}/\{\mathbb{C}\text{-valued exact } \mathcal{C}^1 \text{ 1-forms on } U\}.$$

Define $H_1(U, \mathbb{C})$ as homology classes of cycles which are $\mathbb{C}$-linear combinations of paths. Then Theorem 2.28 and Theorem 8.27 show that the bilinear form

$$H_1(U, \mathbb{C}) \times H^1_{\mathrm{DR}}(U, \mathbb{C}) \to \mathbb{C}, \qquad (\Gamma, \omega) \mapsto \int_\Gamma \omega$$

is well-defined and non-degenerate. In particular we have an injective $\mathbb{C}$-linear map

$$H^1_{\mathrm{DR}}(U, \mathbb{C}) \to \mathrm{Hom}_{\mathbb{C}}(H_1(U, \mathbb{C}), \mathbb{C}), \qquad \omega \mapsto (\Gamma \mapsto \int_\Gamma \omega). \tag{8.29.1}$$

Note that these spaces are not necessarily finite-dimensional (consider the example $U := \mathbb{C} \setminus \mathbb{Z}$).
One can show that (8.29.1) is an isomorphism (e.g., W. Fulton: *Algebraic Topology, A first Course*, Springer (1995), Theorem 15.11).

9 Isolated singularities and meromorphic functions

(A) Laurent series

Definition 9.1. Let $z_0 \in \mathbb{C}$. A *Laurent series in* z_0 is a series of the form

$$(*) \qquad \sum_{n=-\infty}^{\infty} a_n(z-z_0)^n,$$

more precisely, it is a pair of two series

$$\sum_{n=0}^{\infty} a_n(z-z_0)^n \qquad \text{called } \textit{regular part} \text{ (German: } \textit{Nebenteil}\text{)}$$

$$\sum_{n=-1}^{-\infty} a_n(z-z_0)^n := \sum_{n=1}^{\infty} a_{-n}(z-z_0)^{-n} \qquad \text{called } \textit{principal part} \text{ (German: } \textit{Hauptteil}\text{)}.$$

Let $A \subseteq \mathbb{C}$ be a subset. If these two series converge absolutely for $z \in A$ (resp. converges locally uniformly in z on A) we say that (*) *converges absolutely* (resp. *converges locally uniformly*) *on* A. If that is the case, we also denote by $\sum_{n=-\infty}^{\infty} a_n(z-z_0)^n$ the sum $\sum_{n=0}^{\infty} a_n(z-z_0)^n + \sum_{n=-1}^{-\infty} a_n(z-z_0)^n$.

Proposition 9.2. *Let $z_0 \in \mathbb{C}$, let $0 \le r < R \le \infty$, and let $A := \{\, z \in \mathbb{C} \;;\; r < |z-z_0| < R \,\}$ be the annulus centered at z_0 with inner radius r and outer radius R. Let $f\colon A \to \mathbb{C}$ be a holomorphic function. Then f has a Laurent series expansion*

$$f(z) = \sum_{n=-\infty}^{\infty} a_n(z-z_0)^n$$

which converges absolutely and locally uniformly on A. For all $\rho \in \mathbb{R}$ with $r < \rho < R$ and for all $n \in \mathbb{Z}$ we have

$$a_n = \frac{1}{2\pi i} \int_{\partial B_\rho(z_0)} \frac{f(z)}{(z-z_0)^{n+1}}\, dz \tag{9.2.1}$$

and the Cauchy inequalities

$$|a_n| \le \rho^{-n} \|f\|_{\partial B_\rho(z_0)}. \tag{9.2.2}$$

We will see: Regular part converges on $\{\, z \in \mathbb{C} \;;\; |z-z_0| < R \,\}$, and the principal part converges on $\{\, z \in \mathbb{C} \;;\; |z-z_0| > r \,\}$.

Proof. We may assume that $z_0 = 0$. Choose r' and R' such that $r < r' < R' < R$. Then the loops $\partial B_{r'}(0)$ and $\partial B_{R'}(0)$ are homologous in A, hence $\Gamma := \partial B_{R'}(0) - \partial B_{r'}(0) \in B_1(A, \mathbb{Z})$. As $z \mapsto \frac{f(z)}{z^{n+1}}$ is holomorphic in A, Cauchy's theorem shows that (9.2.1) is independant of ρ.

Moreover, Theorem 8.26 gives for w with $r' < |w| < R'$:

$$(*) \qquad \begin{aligned} f(w) = W(\Gamma; w) f(w) &= \frac{1}{2\pi i} \int_{\Gamma} \frac{f(z)}{z-w}\, dz \\ &= \underbrace{\frac{1}{2\pi i} \int_{\partial B_{R'}(0)} \frac{f(z)}{z-w}\, dz}_{=: f_{\mathrm{reg}}(w)} \; \underbrace{- \frac{1}{2\pi i} \int_{\partial B_{r'}(0)} \frac{f(z)}{z-w}\, dz}_{=: f_{\mathrm{princ}}(w)} . \end{aligned}$$

By homotopy invariance f_{reg} is independent of R' as long as $|w| < R' < R$. Therefore f_{reg} can be extended holomorphically to $B_R(0)$. The same argument shows that f_{princ} can be extended holomorphically to $\{\, w \in \mathbb{C} \;;\; |w| > r \,\}$.
By the Cauchy formula for discs (Theorem 5.8) we get $f_{\mathrm{reg}}(w) = \sum_{n=0}^{\infty} a_n w^n$ with a_n as in (9.2.1) for $n \geq 0$.
To handle $f_{\mathrm{princ}}(w)$ we choose r' with $r < r' < |w|$ and write

$$z - w = -w(1 - \frac{z}{w}).$$

For $z \in \partial B_{r'}(0)$ we have $|z/w| < 1$, so the geometric series

$$\frac{1}{z-w} = -\frac{1}{w} \frac{1}{1 - z/w} = -\frac{1}{w} \sum_{n=0}^{\infty} \left(\frac{z}{w}\right)^n = -\sum_{n=-1}^{-\infty} \frac{w^n}{z^{n+1}}.$$

converges. We can exchange the order of summation and integration and obtain

$$f_{\mathrm{princ}}(w) = \sum_{n=-1}^{\infty} \underbrace{\left(\frac{1}{2\pi i} \int_{\partial B_{r'}(0)} \frac{f(z)}{z^{n+1}}\, dz \right)}_{=: a_n} w^n.$$

Finally, (9.2.2) follows from (9.2.1) by using the standard estimate of path integrals. □

Example 9.3. Consider the holomorphic function

$$f \colon \mathbb{C} \setminus \{0, 1\} \to \mathbb{C}, \qquad f(z) = \frac{1}{z(z-1)}.$$

(1) The Laurent series expansion in 0 on the annulus $\{\, z \in \mathbb{C} \;;\; 0 < |z| < 1 \,\}$ is given by

$$f(z) = \frac{1}{z-1} - \frac{1}{z} = \underbrace{-\frac{1}{z}}_{\text{principal part}} + \underbrace{(-\sum_{n=0}^{\infty} z^n)}_{\text{regular part}} .$$

(2) On the annulus $\{\, z \in \mathbb{C} \;;\; |z| > 1 \,\}$ the Laurent series expansion in 0 is given by

$$f(z) = \frac{1}{z} \frac{1}{z} \frac{1}{1 - 1/z} \overset{|1/z|<1}{=} \frac{1}{z^2} \sum_{n=0}^{\infty} \left(\frac{1}{z}\right)^n = \sum_{n=-2}^{-\infty} z^n.$$

(B) Isolated singularities

Definition 9.4. Let $U \subseteq \mathbb{C}$ be open, $z_0 \in U$. If $f\colon U \setminus \{z_0\} \to \mathbb{C}$ is holomorphic, then z_0 is called *isolated singularity of f*.

Theorem and Definition 9.5. *Let $U \subseteq \mathbb{C}$ be open, $z_0 \in U$ and let $f\colon U \setminus \{z_0\} \to \mathbb{C}$ be holomorphic. Let $r \in \mathbb{R}^{>0}$ with $B_r(z_0) \subseteq U$, and let*

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n \tag{*}$$

be the Laurent series expansion of f on the annulus $B_r(z_0) \setminus \{z_0\}$. Then exactly one of the following cases occur.

(a) *The following equivalent conditions are satisfied.*
  (i) *$a_n = 0$ for all $n < 0$, i.e., the principal part of (*) is zero.*
  (ii) *There exists a (necessarily unique) holomorphic function $\tilde{f}\colon U \to \mathbb{C}$ such that $\tilde{f}_{|U\setminus\{z_0\}} = f$.*
  (iii) *f is bounded in some neighborhood of z_0.*
  In this case z_0 is called a removable singularity of f.

(b) *The following equivalent conditions are satisfied.*
  (i) *There exists $k \in \mathbb{N}$ such that $a_{-k} \neq 0$ and $a_n = 0$ for all $n < -k$.*
  (ii) *There exists $k \in \mathbb{N}$ and a holomorphic function $h\colon U \to \mathbb{C}$ with $h(z_0) \neq 0$ such that*
  $$f(z) = (z - z_0)^{-k} h(z) \qquad \text{for all } z \in U \setminus \{z_0\}.$$
  (iii) *One has $\lim_{z \to z_0} |f(z)| = \infty$.*
  Moreover the integers k in (i) and in (ii) are equal.
  In this case z_0 is called a pole of f. *The integer $k \in \mathbb{N}$ is called the* order of the pole z_0.

(c) *The following equivalent conditions are satisfied* (Casorati-Weierstraß)
  (i) *There exist infinitely many $n \in \mathbb{Z}^{<0}$ with $a_n \neq 0$.*
  (ii) *For every $w_0 \in \mathbb{C}$ there exists a sequence $(z_n)_n$ in $U\setminus\{z_0\}$ such that $\lim_n z_n = z_0$ and $\lim_n f(z_n) = w_0$.*
  In this case z_0 is called an essential singularity of f.

Proof. We may assume that $z_0 = 0$. Clearly exactly one of the conditions (a), (b), or (c) is satisfied.
(a). "(i) $\Leftrightarrow$ (ii)": Clear, as holomorphic functions are analytic.
"(ii) $\Leftrightarrow$ (iii)": Riemann extension theorem (Theorem 5.13).
(b). "(i) $\Leftrightarrow$ (ii)": Condition (i) means that the Laurent series expansion is of the form

$$f(z) = a_{-k} z^{-k} + a_{-k+1} z^{-k+1} + \dots + a_{-1} z^{-1} + \sum_{n=0}^{\infty} a_n z^n$$

with $a_{-k} \neq 0$. Hence (i) and (ii) are equivalent.
"(ii) $\Rightarrow$ (iii)": Clear.

"(iii) $\Rightarrow$ (ii)": We have to show that 0 is a removable singularity of $z \mapsto z^k f(z)$. For this we may make U smaller and can assume that $f(z) \neq 0$ for $z \in U \setminus \{0\}$. Then $g := 1/f \colon U \setminus \{0\} \to \mathbb{C}$ is holomorphic and nonzero, and $\lim_{z\to 0} |f(z)| = \infty$ implies $\lim_{z\to 0} |g(z)| = 0$. By the Riemann extension theorem, g can be extended to a holomorphic function on U, again called g. After again making U smaller, we may assume that the power series expansion $g(z) = \sum_{n=0}^{\infty} b_n z^n$ of g in 0 converges on U. Set $k := \inf\{\, n \in \mathbb{N}_0 \ ; \ b_n \neq 0 \,\}$. Then

$$g(z) = z^k \tilde{h}(z),$$

where $\tilde{h}$ is holomorphic on U and $\tilde{h}(0) \neq 0$. Thus $\tilde{h}$ is nonzero on U and hence $h := 1/\tilde{h}$ is holomorphic on U with $h(z) = z^k f(z)$.
(c). "(ii) $\Rightarrow$ (i)": If (ii) is satisfied, z_0 cannot be a removable singularity or a pole as we have already shown. Hence (i) has to be satisfied.
"(i) $\Rightarrow$ (ii)": Assume that (ii) does not hold, i.e., there exist $w_0 \in \mathbb{C}$ and $\rho, \varepsilon > 0$ such that $|f(z) - w_0| \geq \varepsilon$ for all z with $0 < |z| < \rho$. The function

$$g \colon \{\, z \in \mathbb{C} \ ; \ 0 < |z| < \rho \,\} \to \mathbb{C}, \qquad g(z) := \frac{1}{f(z) - w_0}$$

is thus holomorphic and bounded by $1/\varepsilon$. By the Riemann extension theorem, g can be extended holomorphically to $B_\rho(0)$. But then $f = w_0 + 1/g$ has a removable singularity (if $g(0) \neq 0$) or a pole (if $g(0) = 0$) at 0. $\square$

Definition and Remark 9.6. Let $U \subseteq \mathbb{C}$ be open, $z_0 \in U$. Let $f \colon U \setminus \{z_0\} \to \mathbb{C}$ be holomorphic and let $f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$ be the Laurent series expansion of f in z_0. Assume that z_0 is a removable singularity or a pole of f. Then

$$\operatorname{ord}_{z_0}(f) := \inf\{\, k \in \mathbb{Z} \ ; \ a_k \neq 0 \,\} \in \mathbb{Z} \cup \{\infty\}$$

is called the *order of f in z_0*.

(1) Assume $\operatorname{ord}_{z_0}(f) < \infty$. Then

$$\begin{aligned} \operatorname{ord}_{z_0}(f) &= \sup\{\, l \in \mathbb{Z} \ ; \ z \mapsto \frac{f(z)}{(z-z_0)^l} \text{ has a removable singularity in } z_0 \} \\ &= \text{unique integer } k \in \mathbb{Z} \text{ such that } \lim_{z\to z_0} \frac{f(z)}{(z-z_0)^k} \text{ exists in } \mathbb{C} \setminus \{0\}. \end{aligned}$$

(2) f has a pole in z_0 if and only if $\operatorname{ord}_{z_0}(f) < 0$ and in this case the order of the pole is $-\operatorname{ord}_{z_0}(f)$.

Example 9.7.

(1) The holomorphic function $\mathbb{C}^\times \to \mathbb{C}$, $z \mapsto \exp(\frac{1}{z})$ has an essential singularity in 0: Its Laurent series expansion in 0 is

$$\sum_{n=-\infty}^{0} \frac{z^n}{(-n)!}.$$

(2) Let $\log\colon G := \mathbb{C} \setminus \{\, z \in \mathbb{R} \;;\; z < 0 \,\} \to \mathbb{C}$ be the principal branch of the logarithm on G (Definition 6.6). Then its power series expansion at $z_0 = 1$ is given by

$$\log(z) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}(z-1)^n$$

for $z \in B_1(1)$. Thus $G \setminus \{1\} \to \mathbb{C}$, $z \mapsto \frac{\log(z)}{z-1}$ has a removable singularity at 1.

(C) Meromorphic functions

Definition 9.8. Let $U \subseteq \mathbb{C}$ be open. A *meromorphic function f on U* is a holomorphic function $f\colon U \setminus P(f) \to \mathbb{C}$, where $P(f) \subseteq U$ is a closed subset such that every $z \in P(f)$ is a pole of f.
We consider a meromorphic function f on U as a map $f\colon U \to \widehat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ by setting $f(z) := \infty$ for $z \in P(f)$.
We denote by $\mathscr{M}(U)$ the set of meromorphic functions on U.

Remark 9.9. Let $U \subseteq \mathbb{C}$ be open.
(1) Every holomorphic function f on U is meromorphic on U (then $P(f) = \emptyset$).
(2) Let f be meromorphic on U. Then $P(f)$ is discrete and closed in U.

Proof. If z is a pole of f, then there exists $z \in W \subseteq U$ open such that $f_{|W \setminus \{z\}}$ is holomorphic (by definition of "pole"). Thus $P(f) \cap W = \{z\}$ which implies $\{z\}$ is open in $P(f)$. Therefore $P(f)$ is discrete. □

(3) Let $(U_i)_{i \in I}$ be an open covering of U, and let $f\colon U \to \widehat{\mathbb{C}}$ be a map. Then f is meromorphic if and only if $f_{|U_i}$ is meromorphic for all $i \in I$ (use that a subset $S \subseteq U$ is closed in U if and only if $S \cap U_i$ is closed in U_i for all $i \in I$).
(4) Let $f, g \in \mathscr{M}(U)$. Define $f + g$ as follows: Note that $P(f) \cup P(g)$ is again closed in U and discrete[8]. For $z \in U \setminus (P(f) \cup P(g))$ we define $(f+g)(z) := f(z) + g(z)$. Looking at the Laurent series expansion one sees that for $z_0 \in P(f) \cup P(g)$
(a) either z_0 is a removable singularity of $f + g$. Then set

$$(f+g)(z_0) := \lim_{z \to z_0} (f+g)(z).$$

(b) or z_0 is a pole of $f + g$. Then set $(f+g)(z_0) := \infty$.
In the same way, one defines the product fg. These definitions make $\mathscr{M}(U)$ into a $\mathbb{C}$-algebra.

[8] X topological space, $Y, Z \subseteq X$ discrete and closed. Then $Y \cup Z$ is discrete and closed in X. Indeed, $Y \cup Z$ is clearly closed. Let $x \in Y \cup Z$. We have to show that there exists $x \in W \subseteq X$ open such that $W \cap (Y \cup Z) = \{x\}$. After possible switching Y and Z we may assume that $x \in Y$. Let $x \in U \subseteq X$ be open such that $U \cap Y = \{x\}$. If $x \in Z$ then there exist $x \in V \subseteq X$ open such that $V \cap Z = \{x\}$ and we may take $W := U \cap V$. If $x \notin Z$, we may take $W := U \cap (X \setminus Z)$ because $X \setminus Z$ is open in X.

Note that the union of two discrete subspaces is not necessarily discrete: $\{\, 1/n \;;\; n \in \mathbb{N} \,\}$ and $\{0\}$ are both discrete subspaces of $\mathbb{R}$, but their union is not discrete.

Proposition 9.10. *Let $U \subseteq \mathbb{C}$ be open. Then a function f is meromorphic on U if and only if for all $z \in U$ there exists $z_0 \in B_r(z_0) \subseteq U$ open such that $f = g/h$, where $g, h\colon B_r(z_0) \to \mathbb{C}$ are holomorphic with $h \neq 0$ (but of course $h(z_0) = 0$ is possible). In this case*

$$\operatorname{ord}_z(f) = \operatorname{ord}_z(g) - \operatorname{ord}_z(h) \qquad \textit{for all } z \in B_r(z_0) \tag{9.10.1}$$

One can show that there exist even globally holomorphic functions $g, h\colon U \to \mathbb{C}$ such that $f = g/h$.

Proof. Let f be meromorphic on U. For all $z_0 \in U$ we may choose $B_r(z_0)$ such that $f(z) = \sum_{n=k}^{\infty} a_n(z - z_0)^n$ on $B_r(z_0) \setminus \{z_0\}$, where $k = \operatorname{ord}_{z_0}(f) \in \mathbb{Z}$. If $k \geq 0$, we set $g = f$ and $h = 1$ on $B_r(z_0)$. If $k < 0$, set $g := \sum_{n=0}^{\infty} a_{n-k}(z - z_0)^n$ and $h = (z - z_0)^{-k}$.
Conversely let $z_0 \in U$ and let g and h as above. Set $d := \operatorname{ord}_{z_0}(g)$ and $e := \operatorname{ord}_{z_0}(h)$. Then $g(z) = (z - z_0)^d \tilde{g}(z)$ and $h(z) = (z - z_0)^e \tilde{h}(z)$, where (for some $r > 0$)

$$\tilde{g}, \tilde{h}\colon B_r(z_0) \to \mathbb{C}$$

holomorphic with $\tilde{g}(z_0) \neq 0 \neq \tilde{h}(z_0)$. By shrinking r we may assume $\tilde{g}(z) \neq 0 \neq \tilde{h}(z)$ for all $z \in B_r(z_0)$. Hence $f(z) = (z - z_0)^{d-e}\tilde{f}(z)$, where $\tilde{f} := \tilde{g}/\tilde{h}$ is holomorphic with $\tilde{f}(z_0) \neq 0$. Therefore f has an isolated singularity of order $d - e$ at z_0 and thus is meromorphic on $B_r(z_0)$. As z_0 was arbitrary, this shows that f is meromorphic on U (Remark 9.9 (3)). □

Example 9.11. The *tangens function* $\tan := \frac{\sin}{\cos}$ is a meromorphic function on $\mathbb{C}$. For $z \in \mathbb{C}$ we have (see Exercise 23)

$$\begin{aligned} \operatorname{ord}_z(\sin) \neq 0 \Leftrightarrow \sin(z) = 0 \quad &\Leftrightarrow \quad z \in \pi\mathbb{Z} := \{\, \pi k \ ; \ k \in \mathbb{Z} \,\}, \\ \operatorname{ord}_z(\cos) \neq 0 \Leftrightarrow \cos(z) = 0 \quad &\Leftrightarrow \quad z \in \frac{\pi}{2} + \pi\mathbb{Z} := \{\, \frac{\pi}{2} + \pi k \ ; \ k \in \mathbb{Z} \,\}. \end{aligned}$$

For $z \in \pi\mathbb{Z}$ we have $\sin'(z) = \cos(z) \neq 0$ and thus $\operatorname{ord}_z(\sin) = 1$. Analogously $\operatorname{ord}_z(\cos) = 1$ for $z \in \frac{\pi}{2} + \pi\mathbb{Z}$. Therefore:

$$\operatorname{ord}_z(\tan) = \begin{cases} 1, & z \in \pi\mathbb{Z}; \\ -1, & z \in \frac{\pi}{2} + \pi\mathbb{Z}; \\ 0, & \text{otherwise.} \end{cases}$$

(D) Meromorphic functions and the Riemann sphere

We add ∞ to $\mathbb{C}$ and obtain a "complex manifold". We first define $\widehat{\mathbb{C}}$ as topological space.

Definition and Remark 9.12. We define $\widehat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$, where ∞ is some element not contained in $\mathbb{C}$.
Topology of $\widehat{\mathbb{C}}$: A subset U of $\widehat{\mathbb{C}}$ is called open if either $U \subseteq \mathbb{C}$ and U is open in $\mathbb{C}$ or if $\infty \in U$ and $\widehat{\mathbb{C}} \setminus U$ is a compact subspace of $\mathbb{C}$. It is easy to check that this defines a topology on $\widehat{\mathbb{C}}$.

For a sequence $(z_n)_n$ of complex numbers we then have:

$$
\begin{aligned}
&\lim_{n\to\infty} z_n = \infty \quad \text{in } \widehat{\mathbb{C}} \\
\Leftrightarrow\ & \forall\, K \subseteq \mathbb{C} \text{ compact} : \{\, n \ ;\ z_n \in K \,\} \text{ is finite} \\
\Leftrightarrow\ & \forall\, R \in \mathbb{R}^{>0} : \{\, n \ ;\ z_n \in \overline{B_R(0)} \,\} \text{ is finite} \\
\Leftrightarrow\ & \lim_{n\to\infty} |z_n| = \infty.
\end{aligned}
$$

Example 9.13. The function

$$
i \colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}, \quad z \mapsto 1/z := \begin{cases} 1/z, & z \neq 0, \infty; \\ \infty, & z = 0; \\ 0, & z = \infty \end{cases}
$$

is a homeomorphism with $i^{-1} = i$.

We imagine $\widehat{\mathbb{C}}$ as a sphere:

Remark 9.14. Consider $\mathbb{R}^3$ with coordinates x_1, x_2, and x_3, and identify $\mathbb{C}$ with the (x_1, x_2)-plane by setting $\mathbb{C} \ni z = x_1 + ix_2$. Let

$$
S^2 := \{\, (x_1, x_2, x_3) \in \mathbb{R}^3 \ ;\ x_1^2 + x_2^2 + x_3^2 = 1 \,\}
$$

be the 2-sphere, let $N := (0, 0, 1) \in S^2$ be its "north pole", and let $\varphi \colon S^2 \setminus \{N\} \to \mathbb{C}$ be the stereographic projection, i.e. $\varphi(x)$ is the point at which the line connecting N and x intersects $\mathbb{C}$. We obtain a homeomorphism

$$
\varphi \colon S^2 \setminus \{N\} \to \mathbb{C}, \qquad (x_1, x_2, x_3) \mapsto \frac{1}{1 - x_3}(x_1 + ix_2)
$$

whose inverse is given by $x + iy \mapsto (x^2 + y^2 + 1)^{-1}(2x, 2y, x^2 + y^2 - 1)$.
We extend φ to a bijection

$$
\hat{\varphi} \colon S^2 \to \widehat{\mathbb{C}}
$$

by setting $\hat{\varphi}(N) := \infty$. Then it is easy to check that $\hat{\varphi}$ is a homeomorphism.
In particular we see that $\widehat{\mathbb{C}}$ is compact and path-connected (because S^2 has these properties). In fact, it is even simply connected (Exercise 40).

We consider $\widehat{\mathbb{C}}$ as "complex manifold of complex dimension 1 with an atlas consisting of the two charts"

$$
\begin{aligned}
\Phi_0 \colon U_0 := \widehat{\mathbb{C}} \setminus \{\infty\} &\xrightarrow{\sim} \mathbb{C}, \qquad \Phi_0(z) = z; \\
\Phi_1 \colon U_1 := \widehat{\mathbb{C}} \setminus \{0\} &\xrightarrow{\sim} \mathbb{C}, \qquad \Phi_1(z) = 1/z.
\end{aligned}
$$

More precisely:

Definition 9.15. Let $U \subseteq \widehat{\mathbb{C}}$ be open. A continuous map $f\colon U \to \widehat{\mathbb{C}}$ is called *holomorphic* if for all $i, j = 0, 1$ the compositions

$$\underbrace{\Phi_i(U \cap U_i \cap f^{-1}(U_j))}_{\subseteq \mathbb{C} \text{ open}} \xrightarrow{\Phi_i^{-1}} U \cap U_i \cap f^{-1}(U_j) \xrightarrow{f} U_j \xrightarrow{\Phi_j} \mathbb{C}$$

are holomorphic.
Let $V \subseteq \widehat{\mathbb{C}}$ be open. A bijective map $f\colon U \to V$ is called *biholomorphic* if $f\colon U \to \widehat{\mathbb{C}}$ and $f^{-1}\colon V \to \widehat{\mathbb{C}}$ are holomorphic.

Example 9.16. The map $\widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$, $z \mapsto 1/z$ is biholomorphic.

Proposition 9.17. *Let $G \subseteq \widehat{\mathbb{C}}$ be open and connected. Choose $V_0, V_1 \subseteq G$ open and connected such that $G = V_0 \cup V_1$ with $\infty \notin V_0$ and $0 \notin V_1$. Let $f\colon G \to \widehat{\mathbb{C}}$ be continuous such that there exists $z \in G$ with $f(z) \neq \infty$.*
Then f is a holomorphic map if and only if the two maps

$$\begin{aligned} f_0\colon V_0 \to \widehat{\mathbb{C}}, \qquad & z \mapsto f(z), \\ f_1\colon \{\, \frac{1}{z} \in \mathbb{C} \;;\; z \in V_1 \,\} \to \widehat{\mathbb{C}}, \qquad & w \mapsto f(\frac{1}{w}) \end{aligned}$$

are meromorphic.

Proof. By definition, f is holomorphic if and only if f_i is holomorphic (in the sense of Definition 9.15). Thus it suffices to show the following proposition. □

Proposition 9.18. *Let $G \subset \mathbb{C}$ be open and connected, $f\colon G \to \widehat{\mathbb{C}}$ be a map such that there exists $z \in G$ with $f(z) \neq \infty$. Then the following assertions are equivalent:*
(i) *f is meromorphic.*
(ii) *f is a holomorphic map in the sense of Definition 9.15.*

Proof. "(i) ⇒ (ii)". Clearly, $f_{|U \setminus P(f)}$ is holomorphic. Let $z_0 \in P(f)$. Then Theorem 9.5 (b) shows $\lim_{z \to z_0} f(z) = \infty$. Hence f is at least continuous in z_0. Moreover, there exists $z_0 \in W \subseteq U$ open such that $f(z) \notin \{0, \infty\}$ for all $z \in W$. Then $g\colon W \setminus \{z_0\} \to \mathbb{C}$, $g(z) = 1/f(z)$ is holomorphic with $\lim_{z \to z_0} g(z) = 0$. Hence g can be extended holomorphically to z_0. This shows that f is a holomorphic map on W in the sense of Definition 9.15.
"(ii) ⇒ (i)". Set $P(f) := \{\, z \in U \;;\; f(z) = \infty \,\}$. Then $f_{|U \setminus P(f)}\colon U \setminus P(f) \to \mathbb{C}$ is holomorphic by Definition 9.15. As f is continuous, $P(f)$ is closed in U and $\lim_{z \to z_0} f(z) = \infty$ for all $z_0 \in P(f)$. Hence it remains to show that $P(f)$ is discrete in U (⇒ the points of $P(f)$ are poles ⇒ f is meromorphic).
Choose $z_0 \in W \subseteq U$ open such that $f(z) \neq 0$ for all $z \in W$. Then $g\colon W \to \mathbb{C}$, $z \mapsto 1/f(z)$ is holomorphic and $P(f) \cap W = \{\, z \in W \;;\; g(z) = 0 \,\}$. Now $f \neq \infty$ implies $g \neq 0$. and hence $\{\, z \in W \;;\; g(z) = 0 \,\}$ is discrete by Proposition 5.23. □

10 Calculus of Residues

(A) The residue theorem

Remark 10.1. Let $U \subseteq \mathbb{C}$ be open, $z_0 \in U$, $f\colon U \setminus \{z_0\}$ holomorphic. Let

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$

be its Laurent series expansion at z_0. Choose $r \in \mathbb{R}^{>0}$ such that $\overline{B_r(z_0)} \subseteq U$. Then the Laurent series converges on some $B_R(z_0)$ for some $R > r$ and we have

$$\int\limits_{\partial B_r(z_0)} f(z)\,dz = \sum_{n=-\infty}^{\infty} a_n \int\limits_{\partial B_r(z_0)} (z - z_0)^n\,dz,$$

where we may interchange sum and integral because the series converges locally uniformly. But $(z - z_0)^n$ has a primitive on $U \setminus \{z_0\}$ for $n \neq -1$, namely $\frac{1}{n+1}(z - z_0)^{n+1}$. Therefore $\int_\gamma (z - z_0)^n\,dz = 0$ for every loop γ in $U \setminus \{z_0\}$ and for $n \neq -1$. Hence we see that

$$\int\limits_{\partial B_r(z_0)} f(z)\,dz = a_{-1} \int\limits_{\partial B_r(z_0)} (z - z_0)^{-1}\,dz = 2\pi i a_{-1}.$$

This also follows from (9.2.1).

Definition 10.2. Let $U \subseteq \mathbb{C}$ be open, $z_0 \in U$, $f\colon U \setminus \{z_0\} \to \mathbb{C}$ holomorphic with Laurent series $f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$. Then

$$\operatorname{res}_{z_0}(f) := a_{-1} \overset{(9.2.1)}{=} \frac{1}{2\pi i} \int\limits_{\partial B_r(z_0)} f(z)\,dz$$

is called the *residue of f at z_0*. Here we choose $r \in \mathbb{R}^{>0}$ such that $\overline{B_r(z_0)} \subseteq U$.

For the following theorem we make the following remark. Let $U \subseteq \mathbb{C}$ be open, let $S \subseteq U$ be discrete and closed, and let Γ be a 1-cycle in U which is null-homologous in U (i.e. $W(\Gamma; u) = 0$ for all $u \in \mathbb{C} \setminus U$) such that $S \cap \{\Gamma\} = \emptyset$. Then there are only finitely many $z \in S$ with $W(\Gamma; z) \neq 0$. Indeed

$$I := \overline{\{\, z \in \mathbb{C} \setminus \{\Gamma\} \;;\; W(\Gamma; z) \neq 0 \,\}}$$

is compact (Remark 8.23) and contained in U because Γ is null-homologous in U (Exercise 34). As S is discrete and closed in U, $S \cap I$ is a closed and discrete subspace of I. Therefore it is a discrete and compact space and hence finite.

Theorem 10.3 (Residue theorem). *Let $U \subseteq \mathbb{C}$ be open, let $S \subseteq U$ be discrete and closed, let $f\colon U \setminus S \to \mathbb{C}$ be holomorphic. Let Γ be a 1-cycle in U which is null-homologous in U such that $S \cap \{\Gamma\} = \emptyset$. Then*

$$\int\limits_{\Gamma} f(z)\,dz = 2\pi i \sum_{z \in S} W(\Gamma; z) \operatorname{res}_z(f).$$

Recall that a 1-cycle in U is automatically null-homologous if U is simply connected.

Proof. Let $S' = \{ z \in S \ ; \ W(\Gamma; z) \neq 0 \}$. Example 8.25 ahows that Γ is homologous to $\sum_{z \in S'} W(\Gamma, z) \partial B_\varepsilon(z)$ with $\varepsilon > 0$ such that $\overline{B_\varepsilon(z)} \subseteq U$ and $\overline{B_\varepsilon(z)} \cap \overline{B_\varepsilon(z')} = \emptyset$ for $z, z' \in S'$ with $z \neq z'$. Hence

$$\begin{aligned} \int_\Gamma f(z)\,dz &\overset{8.27}{=} \sum_{z \in S} W(\Gamma; z) \int_{\partial B_\varepsilon(z)} f(z)\,dz \\ &\overset{10.2}{=} 2\pi i \sum_{z \in S} W(\Gamma; z) \operatorname{res}_z(f) \end{aligned}$$

□

Remark 10.4. Let $U \subseteq \mathbb{C}$ be open, $z_0 \in U$, $f, g \colon U \setminus \{z_0\} \to \mathbb{C}$ be holomorphic,

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n, \qquad g(z) = \sum_{n=-\infty}^{\infty} b_n (z - z_0)^n$$

their Laurent series expansions.

(1) For $a \in \mathbb{C}$ one has

$$\operatorname{res}_{z_0}(af + g) = a \operatorname{res}_{z_0}(f) + \operatorname{res}_{z_0}(g).$$

(2) Assume that f and g are meromorphic on U, i.e. there exist only finitely many $r < 0$ with $a_r \neq 0$ or $b_r \neq 0$. The Laurent series converge absolutely, we have by Cauchy's formula

$$(*) \qquad f(z)g(z) = \sum_{n=-\infty}^{\infty} c_n (z - z_0)^n, \qquad c_n = \sum_{\substack{k,l \in \mathbb{Z} \\ k+l=n}} a_k b_l,$$

where the sum is finite because of our assumption. In particular:

$$\operatorname{res}_{z_0}(fg) = \sum_{k+l=-1} a_k b_l.$$

(3) Assume that f has a pole of order 1 at z_0 and that g is holomorphic in z_0 (more precisely, g can be extended holomorphically into z_0). Then (2) shows

$$\operatorname{res}_{z_0}(fg) = a_{-1} b_0 = g(z_0) \operatorname{res}_{z_0}(f).$$

(4) Assume that f has a zero of order 1 in z_0 and that g is holomorphic in z_0. Then the -1-st coefficient of the Laurent series of $1/f$ is $1/a_1 = 1/f'(z_0)$ by (*), hence by (3):

$$\operatorname{res}_{z_0} \frac{g}{f} = \frac{g(z_0)}{f'(z_0)}$$

Example 10.5. (1)

$$\operatorname{res}_{\pi/2} \tan(z) = \frac{\sin(\pi/2)}{\cos'(\pi/2)} = -1.$$

(2)

$$\frac{\cos(z)}{z^2} = \frac{1}{z^2}\left(1 - \frac{z^2}{2} + \dots\right) = \frac{1}{z^2} - \frac{1}{2} + \text{terms of higher order.}$$

Hence $\operatorname{res}_0(\frac{\cos(z)}{z^2}) = 0$.

(B) Counting zeros

Proposition 10.6. *Let $U \subseteq \mathbb{C}$ be open, $f\colon U \to \widehat{\mathbb{C}}$ meromorphic, $z_0 \in U$. If $\operatorname{ord}_{z_0}(f) < \infty$, then*

$$\operatorname{ord}_{z_0}(f) = \operatorname{res}_{z_0} \frac{f'}{f}.$$

Proof. Exercise □

Proposition 10.7. *Let $U \subseteq \mathbb{C}$ be open, $f\colon U \to \widehat{\mathbb{C}}$ meromorphic, set*

$$Z(f) = \{\, z \in \mathbb{C} \ ; \ f(z) = 0 \,\}, \qquad P(f) := \{\, z \in \mathbb{C} \ ; \ f(z) = \infty \,\}.$$

Let Γ be a null-homologous cycle in U such that $\{\Gamma\} \cap (P(f) \cup Z(f)) = \emptyset$. Then

$$\frac{1}{2\pi i} \int\limits_{\Gamma} \frac{f'}{f}\, dz = \sum_{u \in U} W(\Gamma; u) \operatorname{ord}_u(f) = \sum_{u \in P(f) \cup Z(f)} W(\Gamma; u) \operatorname{ord}_u(f).$$

Note that $W(\Gamma; u) = 0$ for all but finitely many $u \in P(f) \cup Z(f)$ because $P(f) \cup Z(f)$ is discrete and closed in U (see the remark before the Residue theorem).

Proof. Residue theorem + Proposition 10.6. □

(C) Limits and fibers

Definition 10.8. Let $G \subseteq \mathbb{C}$ be a domain, $f\colon G \to \widehat{\mathbb{C}}$ meromorphic. Assume that f is not constant. For $w \in \mathbb{C}$ we set

$$N_f(w) := \sum_{\substack{z \in G \\ f(z) = w}} \operatorname{ord}_z(f(z) - w) \in \mathbb{N}_0 \cup \{\infty\}.$$

In other words: $N_f(w)$ is the number of $z \in G$ with $f(z) = w$ (w-places) with multiplicity.

Proposition 10.9. *Let $G \subseteq \mathbb{C}$ be a domain, and let $(f_n)_n$ be a sequence of holomorphic functions $f_n\colon G \to \mathbb{C}$ that converges locally uniformly against $f\colon G \to \mathbb{C}$. Let $k \in \mathbb{N}_0$ and $w \in \mathbb{C}$ and assume that $N_{f_n}(w) \leq k$ for all $n \in \mathbb{N}$. Then either $f = w$ constant or $N_f(w) \leq k$.*

Proof. By Weierstraß' theorem of convergence (Theorem 5.14) we know that f is holomorphic. We may assume that $w = 0$ (replace f_n by $f_n - w$). Assume that $f \neq 0$ but there exist $z_1, \ldots, z_m \in G$ with $\sum_{i=1}^m \operatorname{ord}_{z_i}(f) > k$. The set $Z(f) := \{\, z \in G \ ; \ f(z) = 0 \,\}$ is discrete, hence there exist discs $D_i = B_{r_i}(z_i)$ such that $\overline{D_i} \subseteq G$ and $\overline{D_i} \cap Z(f) = \{z_i\}$ for all i. Now choose $\varepsilon > 0$ such that $\varepsilon < |f(z)|$ for all z in the compact set $\bigcup_{i=1}^m \partial D_i$. As $(f_n)_n$ converges uniformly on compact sets towards f, there exists $n \in \mathbb{N}$ such that

$$|f(z) - f_n(z)| < \varepsilon < |f(z)|$$

for all $z \in \bigcup_{i=1}^m \partial D_i$. By Rouché's theorem, f and f_n have the same number of zeros in $\bigcup_{i=1}^m D_i$ (with multiplicity). Contradiction. □

Corollary 10.10. *Let $G \subseteq \mathbb{C}$ be a domain, and let $(f_n)_n$ be a sequence of holomorphic functions $f_n\colon G \to \mathbb{C}$ that converges locally uniformly against $f\colon G \to \mathbb{C}$. If all f_n are injective, then f is either injective or constant.*

Proof. 1st proof. This follows from Proposition 10.9 because

$$f \text{ is injective} \quad \Leftrightarrow \quad N_f(w) \le 1 \text{ for all } w \in \mathbb{C}.$$

Here "$\Leftarrow$" is clear. Conversely, let f be injective and assume that there exists $w \in \mathbb{C}$ such that $N_f(w) > 1$. Then $w = f(z_0)$ for some $z_0 \in G$ such that $\mathrm{ord}_{z_0}(f - f(z_0)) > 1$. By restricting to an open nieghborhood of z_0 and by precomposing with a biholomorphic map (under both operations f stays injective), we may assume that $f(z) = f(z_0) + z^r$ with $r = \mathrm{ord}_{z_0}(f - f(z_0)) > 1$ (Theorem 7.3). But this map is clearly not injective. Contradiction.
2nd proof. A different argument goes as follows: f_n is injective if and only if for all $a \in G$ the holomorphic function $g_{n,a} := g_n\colon G \setminus \{a\} \to \mathbb{C}$, $z \mapsto f_n(z) - f_n(a)$ has no zero (i.e. $N_{g_n}(0) = 0$). As $(g_n)_n$ converges locally uniformly to $g_a\colon G \setminus \{a\} \to \mathbb{C}$, $z \mapsto f(z) - f(a)$, Proposition 10.9 implies for all $a \in G$ that either $g_a = 0$ or that g has no zero. If there exists $a \in G$ such that $g_a = 0$, then f is constant and $g_a = 0$ for all $a \in G$. Hence otherwise g_a has no zero for all $a \in G$ and hence f is injective. $\square$

(D) Application: Fourier transform

Motivation: Let G be a locally compact abelian topological group (e.g. $G = (\mathbb{R}, +)$ or $G = (S^1, \cdot)$ with $S^1 = \{\, z \in \mathbb{C} \;;\; |z| = 1 \,\}$). Let μ_G be a translation invariant measure on the Borel σ-algebra $\mathcal{B}(G)$ of G, a so-called Haar measure (e.g. $\mu_{\mathbb{R}} = \lambda^1$ the Lebesgue measure, or μ_{S^1} the image of λ^1 under the map $[0,1] \to S^1$, $x \mapsto \exp(2\pi i x)$).
Let $\hat{G} := \mathrm{Hom}_{\mathrm{Grp}}(G, S^1)$ (Pontryagin dual).
Let $f \in L^1(G, \mathcal{B}(G), \mu_G; \mathbb{C})$. Define its *Fourier transform*

$$\hat{f}(\chi) := \int_G \chi(t) f(t)\, d\mu_G(t)$$

Examples:
(1) For $G = S^1$ one has an isomorphism

$$\mathbb{Z} \xrightarrow{\sim} \hat{G} = \mathrm{Hom}(S^1, S^1), \qquad n \mapsto (z \mapsto z^{-n})$$

and for $f\colon S^1 \to \mathbb{C}$ integrable:

$$\hat{f}(n) = \int_{S^1} t^n f(t)\, d\mu_{S^1} = \int_0^1 e^{2\pi i n t} f(e^{2\pi i t})\, dt, \qquad n \in \mathbb{Z}$$

cf. Proseminar on Fourier analysis.

(2) For $G = \mathbb{R}$ one has an isomorphism

$$\mathbb{R} \mapsto \hat{G} = \mathrm{Hom}(\mathbb{R}, S^1), \qquad \tau \mapsto (t \mapsto e^{-2\pi i\tau t}).$$

and hence for $f\colon \mathbb{R} \to \mathbb{R}$ integrable:

$$\hat{f}(\tau) = \int_{\mathbb{R}} e^{-2\pi i\tau t} f(t)\, d\lambda^1(t), \qquad \tau \in \mathbb{R}.$$

Interpretation: Let $(t \mapsto f(t)) \in L^1(\mathbb{R})$ be a time-dependent function (t measured in seconds). Then $\hat{f}(\tau)$ measures whether the frequency $|\tau|$ (τ measured in hertz) is present in f.

Theorem 10.11. *Let f be meromorphic on $\mathbb{C}$ such that $P(f)$ is finite and $P(f) \cap \mathbb{R} = \emptyset$. Assume that there exists a constant $M \in \mathbb{R}$ such that*

$$|f(z)| \le \frac{M}{|z|}$$

for $|z|$ sufficiently large. Let $\tau \in \mathbb{R}^{<0}$. Then

$$\int_{-\infty}^{\infty} e^{-2\pi i\tau t} f(t)\, dt = 2\pi i \sum_{z \in \mathbb{H}} \mathrm{res}_z(e^{-2\pi i\tau z} f(z)),$$

where $\mathbb{H} := \{\, z \in \mathbb{C} \;;\; \mathrm{Im}(z) > 0 \,\}$ is the upper half plane.

Proof. For simplicity, take $\tau = -1$ (the proof in the general case is the same). Let $A, B \in \mathbb{R}^{>0}$ and set $D := A + B$. By the residue theorem we have for A and B sufficiently large

$$2\pi i \sum_{z \in \mathbb{H}} \mathrm{res}_z(e^{2\pi i z} f(z)) = \int_{\overline{-A,-A+iD,B+iD,B,-A}} e^{2\pi i z} f(z)\, dz.$$

It suffices to show that the integral over the three sides other than the bottom side of this rectangle tend to 0 as A, B tend to infinity. For this we first note that for $x = \mathrm{Re}(z)$ and $y = \mathrm{Im}(z)$ we have

$$e^{2\pi i z} = e^{2\pi i x} e^{-2\pi y}.$$

For the top side of the rectangle we have

$$\begin{aligned}
\Big| -\int_{\overline{-A+iD,B+iD}} e^{2\pi i z} f(z)\, dz\Big| &= \Big|\int_{-A}^{B} e^{2\pi i x} e^{-2\pi D} f(x+iD)\, dx\Big| \\
&\le e^{-2\pi D} \int_{-A}^{B} |f(x+iD)|\, dx \\
&\le e^{-2\pi D} \frac{M}{D}(A+B) = M e^{-2\pi(A+B)} \\
&\overset{A,B\to\infty}{\longrightarrow} 0.
\end{aligned}$$

For the right side we have

$$\begin{aligned}
|\int_{B+iD,B} e^{2\pi iz} f(z)\,dz| &= |\int_0^D e^{2\pi iB} e^{-2\pi y} f(B+iy)\,dy| \\
&\leq \frac{M}{B}\int_0^D e^{-2\pi y}\,dy \\
&= \frac{M}{2\pi B}(1-e^{-2\pi D}) \\
&\overset{A,B\to\infty}{\longrightarrow} 0.
\end{aligned}$$

A similar estimate shows that the integral over the left side tends to 0. $\square$

11 Riemann mapping theorem

Notation: In this section we let $\mathbb{E} := \{\, z \in \mathbb{C} \;;\; |z| < 1 \,\}$ be the open unit disc and $\mathbb{H} := \{\, z \in \mathbb{C} \;;\; \mathrm{Im}(z) > 0 \,\}$ the upper half plane.

(A) Automorphisms of the disc and of the upper half plane

Definition 11.1. Let $U \subseteq \widehat{\mathbb{C}}$ be open. A *biholomorphic automorphism of* U is a biholomorphic map $f\colon U \to U$. We denote by $\mathrm{Aut}_{\mathrm{hol}}(U)$ the set of biholomorphic automorphisms of U, endowed with a group structure by composition:

$$\mathrm{Aut}_{\mathrm{hol}}(U) \times \mathrm{Aut}_{\mathrm{hol}}(U) \to \mathrm{Aut}_{\mathrm{hol}}(U), \qquad (f,g) \to f \circ g.$$

Example 11.2.
(1) For $a, b \in \mathbb{C}$, $a \neq 0$, the maps $\mathbb{C} \to \mathbb{C}$, $z \mapsto az+b$ are biholomorphic automorphism[9].
(2) For all $\varphi \in \mathbb{R}$ the *rotations* $z \mapsto e^{i\varphi}z$ by the angle φ are biholomorphic automorphisms of $B_r(0)$ for all $r > 0$.

Proposition 11.3 (Schwarz lemma). *Let* $f\colon \mathbb{E} \to \mathbb{E}$ *be holomorphic with* $f(0) = 0$. *Then*
(1) $|f(z)| \leq |z|$ *for all* $z \in \mathbb{E}$.
(2) *If there exists* $0 \neq z_0 \in \mathbb{E}$ *with* $|f(z_0)| = |z_0|$, *then* f *is a rotation.*
(3) $|f'(0)| \leq 1$, *and if equality holds, then* f *is a rotation.*

[9] In fact, these are the only biholomorphic automorphisms of $\mathbb{C}$: If $f\colon \mathbb{C} \to \mathbb{C}$ is any automorphism of $\mathbb{C}$, then f is a homeomorphism. Thus for all $R \in \mathbb{R}^{>0}$, $f^{-1}(\overline{B_R(0)})$ is compact. In other words: If we define $\hat{f}\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ by $\hat{f}(z) := f(z)$ for $z \in \mathbb{C}$ and $\hat{f}(\infty) := \infty$, then for every open neighborhood U of ∞ in $\widehat{\mathbb{C}}$, $\hat{f}^{-1}(U)$ is an open neighborhood of ∞. This shows that $\hat{f}$ is continuous in ∞ and hence a holomorphic map $\widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$. By Exercise 41(e), $\hat{f}$ is of the form $z \mapsto \frac{az+b}{cz+d}$ with $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{C})$. As $\hat{f}$ has no pole in $\mathbb{C}$, one has $c = 0$ and hence f is of the form $z \mapsto (a/d) + (b/d)z$.

Proof. *(1).* $f(0) = 0 \Rightarrow \mathrm{ord}_0(f) \geq 1 \Rightarrow \mathrm{ord}_0(f(z)/z) = \mathrm{ord}_0(f) - \mathrm{ord}_0(z) = \mathrm{ord}_0(f) - 1 \geq 0$. Thus $z \mapsto g(z) := f(z)/z$ is holomorphic in $\mathbb{E}$ with $g(0) = f'(0)$. Let $r < 1$. For $z \in \partial B_r(0)$ one has $|g(z)| \leq 1/r$ (because $|f(z)| < 1$). By Corollary 7.11 we have $|g(z)| \leq 1/r$ for all $z \in \overline{B_r(0)}$. Letting $r \to 1$ gives (a).
(2).

$$\begin{aligned} |f(z_0)| = |z_0| &\overset{(1)}{\Rightarrow} |g| \text{ has a maximum in } \mathbb{E} \\ &\overset{7.10}{\Rightarrow} g \text{ is constant} \\ &\Rightarrow f(z) = cz \text{ for some } c \in \mathbb{C} \\ &\overset{|f(z_0)|=|z_0|}{\Rightarrow} |c| = 1. \end{aligned}$$

(3). (1) $\Rightarrow |g(z)| \leq 1$ for all $z \in \mathbb{E}$ and hence $|g(0)| = |f'(0)| \leq 1$. If equality holds, $g = c$ for some constant $c \in \mathbb{C}$ by the maximum principle. Then $|g(0)| = 1 \Rightarrow |c| = 1$. □

Corollary 11.4. *Let $f\colon \mathbb{E} \to \mathbb{E}$ be a biholomorphic automorphism with $f(0) = 0$. Then f is a rotation.*

Proof. $f(0) = 0$ and $f^{-1}(0) = 0$. Applying the Proposition 11.3 (1) to f and f^{-1} implies $|f(z)| = |z|$ for all $z \in \mathbb{E}$. Hence Proposition 11.3 (2) implies that f is a rotation. □

Proposition 11.5. *Let $f\colon \mathbb{E} \to \mathbb{E}$ be a biholomorphic automorphism. Then there exist $\varphi \in \mathbb{R}$ and $\alpha \in \mathbb{E}$ such that*

$$f(z) = e^{i\varphi}\frac{\alpha - z}{1 - \bar{\alpha}z}. \tag{*}$$

Proof. For $\alpha \in \mathbb{E}$ define

$$\psi_\alpha\colon \mathbb{E} \to \mathbb{C}, \qquad \psi_\alpha(z) = \frac{\alpha - z}{1 - \bar{\alpha}z} \tag{11.5.1}$$

(i). Claim: ψ_α is a biholomorphic automorphism $\mathbb{E} \to \mathbb{E}$ (and hence f as in (*) is a biholomorphic automorphism); see Exercise 27 (b).
For $u \in \mathbb{C}$ with $|u| = 1$ write $u = e^{i\theta}$. Then

$$\psi_\alpha(e^{i\theta}) = \frac{\alpha - e^{i\theta}}{e^{i\theta}(e^{-i\theta} - \alpha)} = e^{-i\theta}\frac{w}{\bar{w}}, \qquad \text{with } w := \alpha - e^{i\theta}.$$

Therefore $|\psi_\alpha(u)| = 1$. Hence $\psi_\alpha(z) \in \mathbb{E}$ for all $z \in \mathbb{E}$ (Corollary 7.11).
Moreover ψ_α is its own inverse: For $z \in \mathbb{E}$ we have

$$\begin{aligned} (\psi_\alpha \circ \psi_\alpha)(z) &= \frac{\alpha - \frac{\alpha - z}{1 - \bar{\alpha}z}}{1 - \bar{\alpha}\frac{\alpha - z}{1 - \bar{\alpha}z}} \\ &= \frac{\alpha - |\alpha|^2 z - \alpha + z}{1 - \bar{\alpha}z - |\alpha|^2 + \bar{\alpha}z} \\ &= \frac{(1 - |\alpha|^2)z}{1 - |\alpha|^2} \\ &= z. \end{aligned}$$

(ii). Now let $f\colon \mathbb{E} \to \mathbb{E}$ be a biholomorphic automorphism. Then there exists $\alpha \in \mathbb{E}$ such that $f(\alpha) = 0$. Thus $g := f \circ \psi_\alpha$ is a biholomorphic automorphism of $\mathbb{E}$ with $g(0) = 0$, hence $g(z) = e^{i\varphi} z$ for some $\varphi \in \mathbb{R}$ (Corollary 11.4) and thus

$$f = g \circ \psi_\alpha^{-1} = g \circ \psi_\alpha. \qquad \square$$

Proposition and Definition 11.6. *The map*

$$f\colon \mathbb{H} \to \mathbb{E}, \qquad f(z) := \frac{z-i}{z+i}$$

is biholomorphic. It is called the Cayley map.

Proof. Clearly f is holomorphic. Moreover for $z \in \mathbb{H}$, we have $|z - i| < |z + i|$, thus f maps $\mathbb{H}$ to $\mathbb{E}$. We claim that

$$g\colon \mathbb{E} \to \mathbb{C}, \qquad g(w) := i\frac{1+w}{1-w}$$

yields an inverse of f.
We first show $g(w) \in \mathbb{H}$ for $w \in \mathbb{E}$: Let $w = u + iv$ with $u, v \in \mathbb{R}$. Then

$$\begin{aligned} \operatorname{Im}(g(w)) &= \operatorname{Re}\left(\frac{1+u+iv}{1-u+iv}\right) \\ &= \operatorname{Re}\left(\frac{(1+u+iv)(1-u-iv)}{(1-u)^2+v^2}\right) \\ &= \frac{1-u^2+v^2}{(1+u)^2+v^2} > 0 \end{aligned}$$

since $w \in \mathbb{E}$ and hence $u^2 < 1$. Moreover for $w \in \mathbb{E}$,

$$f(g(w)) = \frac{i(\frac{1+w}{1-w} - 1)}{i(\frac{1+w}{1-w} + 1)} = \frac{1+w-1+w}{1+w+1-w} = w,$$

and similarly $g(f(z)) = z$ for $z \in \mathbb{H}$[10]. $\square$

Remark 11.7. Let $U, V \subseteq \mathbb{C}$ be open, and let $F\colon U \to V$ be biholomorphic. Then

$$F^*\colon \operatorname{Aut}_{\mathrm{hol}}(V) \to \operatorname{Aut}_{\mathrm{hol}}(U), \qquad \alpha \mapsto F^{-1} \circ \alpha \circ F$$

is an isomorphism of groups whose inverse is given by $\beta \mapsto F \circ \beta \circ F^{-1}$.

Theorem 11.8. *For $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{R})$ the map*

$$f_M\colon \mathbb{H} \to \mathbb{C}, \qquad z \mapsto \frac{az+b}{cz+d}$$

[10] One can also show that $\mathrm{GL}_2(\mathbb{C}) \to \operatorname{Aut}_{\mathrm{hol}}(\widehat{\mathbb{C}})$, $M := \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \mapsto (f_M\colon z \mapsto \frac{az+b}{cz+d})$ is a group homomorphism and for $M, N \in \mathrm{GL}_2(\mathbb{C})$ we have $f_M = f_N$ if $M = \lambda N$ for some $\lambda \in \mathbb{C}^\times$. Then the Cayley map is f_M for $M = \left(\begin{smallmatrix} 1 & -i \\ 1 & i \end{smallmatrix}\right)$ with inverse f_N with $N = \frac{1}{2i}\left(\begin{smallmatrix} i & i \\ -1 & 1 \end{smallmatrix}\right)$.

yields a biholomorphic automorphism of $\mathbb{H}$*, and every biholomorphic automorphism of* $\mathbb{H}$ *is of this form. More precisely, the map*

$$\alpha\colon \mathrm{SL}_2(\mathbb{R}) \to \mathrm{Aut}_{\mathrm{hol}}(\mathbb{H}), \qquad M \mapsto f_M$$

is a surjective group homomorphism with kernel $\mu_2 := \{\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right), \left(\begin{smallmatrix}-1&0\\0&-1\end{smallmatrix}\right)\}$.
Finally, for all $z, w \in \mathbb{H}$ *there exists* $M \in \mathrm{SL}_2(\mathbb{R})$ *such that* $f_M(z) = w$.

Therefore we see that $\mathrm{Aut}_{\mathrm{hol}}(\mathbb{H}) \cong \mathrm{PSL}_2(\mathbb{R}) := \mathrm{SL}_2(\mathbb{R})/\mu_2$. The last assertion means that $\mathrm{SL}_2(\mathbb{R})$ acts transitively on $\mathbb{H}$.

Proof. (i). For $M = \left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{R})$ and $z \in \mathbb{H}$ we have

$$(*) \qquad \mathrm{Im}(f_M(z)) = \frac{(ad-bc)\mathrm{Im}(z)}{|cz+d|^2} > 0,$$

hence $f_M(\mathbb{H}) \subseteq \mathbb{H}$.
(ii). For $M, M' \in \mathrm{SL}_2(\mathbb{R})$ one has $f_M \circ f_{M'} = f_{MM'}$ (straightforward calculation). Hence f_M is an automorphism with inverse $f_{M^{-1}}$ and $M \mapsto f_M$ is a group homomorphism $\mathrm{SL}_2(\mathbb{R}) \to \mathrm{Aut}_{\mathrm{hol}}(\mathbb{H})$.
(iii). Let $F\colon \mathbb{H} \to \mathbb{E}$ be the Cayley map, thus $F = f_C$ with $C = \left(\begin{smallmatrix}1&-i\\1&i\end{smallmatrix}\right)$. For $\varphi \in \mathbb{R}$ let $r_\varphi\colon \mathbb{E} \to \mathbb{E}$, $z \mapsto e^{i\varphi}z$ be the rotation by $\varphi \in \mathbb{R}$. Then

$$F^*(r_\varphi) = f_{M_\theta}, \qquad \text{with } M_\theta = \begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix},\ \theta := -\varphi/2$$

by an easy calculation (use $r_\varphi = f_{R_\varphi}$ with $R_\varphi = \left(\begin{smallmatrix}e^{i\varphi}&0\\0&1\end{smallmatrix}\right)$ and $CM_\theta = e^{i\theta}R_\varphi C$, hence $f_C \circ f_{M_\theta} = f_{R_\varphi} \circ f_C$).
(iv). Proof of the last assertion: It suffices to show that for all $z \in \mathbb{H}$ there exists $M = M_z \in \mathrm{SL}_2(\mathbb{R})$ such that $f_M(z) = i$ (then $f_{M_w^{-1}} f_{M_z}(z) = w$ for $z, w \in \mathbb{H}$). Let $c \in \mathbb{R}$ such that $c^2 = \mathrm{Im}(z)/|z|^2$ and set $M_1 := \left(\begin{smallmatrix}0&-c^{-1}\\c&0\end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{R})$. Then (*) shows that $\mathrm{Im}(f_{M_1}(z)) = 1$, say $f_{M_1}(z) = u + i$ with $u \in \mathbb{R}$. For $M_2 = \left(\begin{smallmatrix}1&-u\\0&1\end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{R})$ we have $f_{M_2}(w) = w - u$ in particular

$$f_{M_2M_1}(z) \overset{(ii)}{=} f_{M_2}(f_{M_1}(z)) = u + i - u = i.$$

(v). "$M \mapsto f_M$ is surjective": Let $f \in \mathrm{Aut}_{\mathrm{hol}}(\mathbb{H})$ and let $\beta \in \mathbb{H}$ with $f(\beta) = i$. By (iv) there exists $N \in SL_2(\mathbb{R})$ with $f_N(i) = \beta$. Therefore $g := f \circ f_N$ satisfies $g(i) = i$ and $F \circ g \circ F^{-1} \in \mathrm{Aut}_{\mathrm{hol}}(\mathbb{E})$ fixes 0. Thus $F \circ g \circ F^{-1}$ is a rotation and hence $g = f_{M_\theta}$ for some $\theta \in \mathbb{R}$ by (iii). Therefore $f = f_{M_\theta N^{-1}}$.
(vi). "$\ker(\alpha) = \mu_2$": Clearly $f_M = \mathrm{id}_{\mathbb{H}}$ for $M \in \mu_2$ and hence $\mu_2 \subseteq \ker(\alpha)$. Conversely, let $M = \left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{R})$ with $f_M = \mathrm{id}_{\mathbb{H}}$. (iii) shows that $f_M(i) = i$ implies $a = d$ and $b = -c$. Then an easy calculation shows that $f_M(1+i) = 1+i$ implies $b = c = 0$. Therefore $a = d = \pm 1$ because $M \in \mathrm{SL}_2(\mathbb{R})$. □

(B) The theorem of Arzela-Ascoli

Definition 11.9. Let V be a finite-dimensional $\mathbb{R}$-vector space endowed with some norm $\|\cdot\|$. Let $X \subseteq V$ be open. Let (Y, d) be a metric space, $\mathcal{C}(X, Y) := \{ f \colon X \to Y \ ; \ f \text{ continuous}\}$. Let $\Phi \subseteq \mathcal{C}(X, Y)$ be a subset.

(1) Φ is called *normal* if every sequence in Φ has a subsequence that converges locally uniformly (the limit is not necessarily in Φ).
(2) Φ is called *uniformly bounded on compact subspaces* if for each compact subspace $K \subseteq X$ the subset $\{ f(z) \ ; \ z \in K, f \in \Phi \} \subseteq Y$ is bounded.
(3) Then Φ is called *equicontinuous on compact subspaces* (German: *gleichgradig stetig auf kompakten Teilräumen*) if for each compact subspace $K \subseteq X$ and for all $\varepsilon > 0$ there exists $\delta > 0$ such that for all $z, w \in K$ one has

$$\|z - w\| < \delta \Rightarrow d(f(z), f(w)) < \varepsilon, \qquad \text{for all } f \in \Phi.$$

Theorem 11.10 (Theorem of Arzela-Ascoli). *Let V and Y be a finite-dimensional $\mathbb{R}$-vector spaces and let $X \subseteq V$ be open. Endow V and Y with norms $\|\cdot\|$. Let $\Phi \subseteq \mathcal{C}(X, Y)$ be a subset. If Φ is uniformly bounded and equicontinuous on compact subsets, then Φ is normal.*

There are more general and different variants of this theorem, for instance for an arbitrary compact space X (e.g., Bourbaki: *General topology*, Chap. X, §2) but then one has to be more carefully in the definitions (Definition 11.9).

Proof. Let $(f_n)_n$ be a sequence in Φ. Choose a sequence $(w_j)_{j \in \mathbb{N}}$ in X such that $\{ w_j \ ; \ j \in \mathbb{N} \}$ is dense in X (e.g., identify $V \cong \mathbb{R}^d$ as vector spaces and choose a numbering of $X \cap \mathbb{Q}^d$). By hypothesis, the set $\{ f_n(w_1) \ ; \ n \in \mathbb{N} \}$ is bounded in Y and hence there exists an infinite subset $N_1 \subseteq \mathbb{N}$ such that $(f_n(w_1))_{n \in N_1}$ converges in Y.
Continuing the process we obtain for all $j \in \mathbb{N}$ infinite subsets $N_j \subseteq N_{j-1}$ such that $(f_n(w_j))_{n \in N_j}$ converges in Y. For all $m \in \mathbb{N}$ choose $n_m \in N_m$ such that $n_m \to \infty$ for $m \to \infty$ and set $g_m := f_{n_m} \in \Phi$. Then for all $j \in \mathbb{N}$ the sequence $(g_m(w_j))_{m \geq j}$ is a subsequence of $(f_n(w_j))_{n \in N_j}$ and hence converges for all $j \in \mathbb{N}$.
We claim that $(g_m)_m$ converges locally uniformly. By Remark 2.19 it suffices to show that $(g_m)_m$ converges uniformly on every compact subspace K of X. Let $\varepsilon > 0$. Choose $\delta > 0$ such that for $x, y \in K$ one has

$$\|x - y\| < \delta \Rightarrow \|f(x) - f(y)\| < \varepsilon \qquad \text{for all } f \in \Phi.$$

As $\{ w_j \ ; \ j \in \mathbb{N} \}$ is dense in X, there exists for all $x \in X$ a w_j with $\|x - w_j\| < \delta$; in other words, $X \subseteq \bigcup_{j \in \mathbb{N}} B_\delta(w_j)$. As K is compact, there exists a finite subset $J \subset \mathbb{N}$ such that $K \subseteq \bigcup_{j \in J} B_\delta(w_j)$. Pick $N \in \mathbb{N}$ so large that for $n, m > N$ we have

$$\|g_m(w_j) - g_n(w_j)\| < \varepsilon \qquad \forall\, j \in J.$$

Let $z \in K$, say $z \in B_\delta(w_j)$ for some $j \in J$. Then we have for $n, m > N$:

$$\|g_n(z) - g_m(z)\| \leq \|g_n(z) - g_n(w_j)\| + \|g_n(w_j) - g_m(w_j)\| + \|g_m(w_j) - g_m(z)\| < 3\varepsilon.$$

Hence $(g_{m|K})_m$ is a Cauchy sequence in the $\mathbb{R}$-vector space $\mathcal{C}(K; Y)$ endowed with the supremum norm. As this space is a Banach space, $(g_m)_m$ converges uniformly on K. □

(C) Montel's theorem

Theorem 11.11 (Montel's theorem). *Let $U \subseteq \mathbb{C}$ be open and let $\Phi \subseteq \mathscr{O}(U) = \{ f\colon U \to \mathbb{C} \ ; \ f$ holomorphic$\}$ be a subset which is uniformly bounded. Then Φ is normal.*

Remark 11.12. For real analytic functions the analogue of Montel's theorem is wrong. Consider $f_n\colon (0,1) \to \mathbb{R}$, $f_n(x) = \sin(nx)$. Then $\Phi = \{ f_n \ ; \ n \in \mathbb{N} \}$ is uniformly bounded, but there exists no subsequence of $(f_n)_n$ that converges even pointwise.

Proof of Montel's theorem. By the Theorem of Arzela-Ascoli it suffices to prove that Φ is equicontinuous on compact subspaces. Let $K \subseteq U$ be compact. Choose $r \in \mathbb{R}^{>0}$ such that $B_{3r}(z) \subseteq U$ for all $z \in K$. Let $z, w \in K$ with $|z - w| < r$ and let $\gamma := \partial B_{2r}(w)$. Then for $\zeta \in \{\gamma\}$ one has $|\zeta - w| = 2r$ and $|\zeta - z| \geq r$. Therefore

$$(*) \qquad \left| \frac{1}{\zeta - z} - \frac{1}{\zeta - w} \right| = \frac{|z-w|}{|\zeta - z||\zeta - w|} \leq \frac{|z-w|}{2r^2}.$$

Let K' be the compact set $\{ z \in \mathbb{C} \ ; \ \mathrm{dist}(z, K) \leq 2r \} \subseteq U$. By hypothesis there exists $B \in \mathbb{R}$ such that $|f(z)| \leq B$ for all $f \in \Phi$ and all $z \in K'$.
Then Cauchy integral formula yields for all $z, w \in K$ with $|z-w| < r$ and for all $f \in \Phi$:

$$\begin{aligned} |f(z) - f(w)| &= \left| \frac{1}{2\pi i} \int_\gamma f(\zeta) \left(\frac{1}{\zeta - w} - \frac{1}{\zeta - w} \right) d\zeta \right| \\ &\overset{(*)}{\leq} \frac{1}{2\pi} 4\pi r B \frac{|z-w|}{2r^2} \\ &= C|z-w| \qquad \text{with } C := B/r. \end{aligned}$$

Hence for all $\varepsilon > 0$ we have for $\delta = \min\{r, \varepsilon/C\}$ that for all $z, w \in K$ and $f \in \Phi$:

$$|z - w| < \delta \Rightarrow |f(z) - f(w)| < \varepsilon.$$

Therefore Φ is equicontinuous on K. □

(D) Riemann mapping theorem

Definition 11.13. Two open subsets U and U' of $\mathbb{C}$ are called *biholomorphic equivalent* or *conformal equivalent* if there exists a biholomorphic map $f\colon U \to U'$.

Example 11.14.
(1) We have seen in Proposition 11.6 that $\mathbb{E}$ and $\mathbb{H}$ are biholomorphic equivalent.
(2) The open simply connected sets $\mathbb{C}$ and $\mathbb{E}$ are not biholomorphic equivalent (Liouville's theorem implies that every holomorphic function $\mathbb{C} \to \mathbb{E}$ is constant).
(3) (1) and (2) imply that $\mathbb{C}$ and $\mathbb{H}$ are not biholomorphic equivalent.
(4) The domains $\mathbb{C} \setminus \mathbb{R}^{\leq 0}$ and $\{ z \in \mathbb{C} \ ; \ -\pi < \mathrm{Im}(z) < \pi \}$ are biholomorphc equivalent via the principal branch of logarithm (Example 7.5).

(5) If $G \subseteq \mathbb{C}$ is connected (resp. simply connected) and $U \subseteq \mathbb{C}$ open is biholomorphic equivalent to G, then U is connected (resp. simply connected).

Theorem 11.15 (Riemann mapping theorem). *Let $G \subseteq \mathbb{C}$ be a simply connected domain with $G \neq \mathbb{C}$. Let $z_0 \in G$. Then there exists a unique biholomorphic map $F\colon G \to \mathbb{E}$ such that*

$$F(z_0) = 0 \qquad \textit{and} \qquad F'(z_0) \in \mathbb{R}^{>0}.$$

Corollary 11.16. *Any two simply connected domains $G \subsetneq \mathbb{C}$ and $G' \subsetneq \mathbb{C}$ are biholomorphic equivalent.*

Proof of the Riemann mapping theorem. (i). Unicity:
Let $F_1, F_2\colon G \to \mathbb{E}$ be biholomorphic maps that satisfy the above conditions. Then $H := F_1 \circ F_2^{-1}$ is an automorphism of $\mathbb{E}$ with $H(0) = 0$. Hence $H(z) = e^{i\varphi}z$ for some $\varphi \in \mathbb{R}$ (Corollary 11.4). As $H'(0) = F_1'(0)F_2'(0)^{-1} \in \mathbb{R}^{>0}$ we have $e^{i\varphi} = 1$ and hence $F_1 = F_2$.
(ii). Claim: G is biholomorphic equivalent to an open subset of $\mathbb{E}$ that contains 0.
As $G \neq \mathbb{C}$ there exists $\alpha \in \mathbb{C} \setminus G$. Then $z \mapsto z - \alpha$ is non-zero on G. As G is simply connected, there exists a function

$$L\colon G \mapsto \mathbb{C}$$

with $e^{L(z)} = z - \alpha$ (Proposition 6.4). In particular, L is injective.
Fix $w \in G$. Then there exists $\varepsilon > 0$ such that

$$(*) \qquad B_\varepsilon(L(w) + 2\pi i) \cap L(G) = \emptyset.$$

Otherwise there would exist a sequence $(z_n)_n$ in G such that $\lim_n L(z_n) = L(w) + 2\pi i$. Applying exp we would obtain $\lim_n z_n = w$ because exp is continuous. Hence $\lim_n L(z_n) = L(w)$; contradiction. Now consider

$$F\colon G \to \mathbb{C}, \qquad F(z) := \frac{1}{L(z) - (L(w) + 2\pi i)}.$$

As L is injective, F is injective. Hence $F\colon G \to F(G)$ is biholomorphic (Corollary 7.4). By (*) one has $|F(z)| \leq 1/\varepsilon$ for all $z \in G$, hence $F(G)$ is bounded. We may therefore translate and rescale F to obtain a biholomorphic map with $0 \in F(G) \subseteq \mathbb{E}$.
(iii). By (ii) we may assume that $0 \in G \subseteq \mathbb{E}$. Define

$$\Phi := \{\, f\colon G \to \mathbb{E} \ ;\ f \text{ holomorphic, injective and } f(0) = 0\}.$$

Then $\Phi \neq \emptyset$ because Φ contains the inclusion, and Φ is uniformly bounded because all functions in Φ take only values in $\mathbb{E}$. Moreover, (5.8.2) shows that there exists $C \in \mathbb{R}^{>0}$ with $|f'(0)| \leq C$ for all $f \in \Phi$. Hence

$$s := \sup_{f \in \Phi} |f'(0)|.$$

exists. We will show that there exists $f \in \Phi$ with $|f'(0)| = s$.
Choose a sequence $(f_n)_n$ in Φ such that $\lim_n |f_n'(0)| = s$. By Montel's theorem (Theorem 11.11), this sequence has a subsequence $(f_{n_k})_k$ that converges locally uniformly to some function $f\colon G \to \mathbb{C}$. By Theorem 5.14, f is holomorphic and $\lim_k f_{n_k}' = f'$. In particular $|f'(0)| = s$.
We claim that $f \in \Phi$: Clearly we have $f(0) = 0$. Since $s \geq 1$ (because $z \mapsto z$ is in Φ), f is non-constant and hence injective by Corollary 10.10. By continuity we have $|f(z)| \leq 1$ for all $z \in G$ and from the maximum modulus principle we see that $|f(z)| < 1$ for all $z \in G$. This shows the claim.
(iv). Claim: $f\colon G \to \mathbb{E}$ is surjective ($\overset{7.4}{\Rightarrow}$ f is biholomorphic).
Assume there exists $\alpha \in \mathbb{E} \setminus f(G)$. Consider the automorphism (11.5.1)

$$\psi_\alpha\colon \mathbb{E} \to \mathbb{E}, \qquad \psi_\alpha(z) = \frac{\alpha - z}{1 - \bar{\alpha} z}.$$

Then $G' := \psi_\alpha(f(G)) \subseteq \mathbb{E}$ is simply connected with $0 \notin G'$. Thus there exists a holomorphic square root function $r_2\colon G' \to \mathbb{E}$, i.e., r_2 satisfies $r_2(z)^2 = z$ for all $z \in G'$ (Remark 6.9). Set $\beta := r_2(\alpha)$ and define

$$F := \psi_\beta \circ r_2 \circ \psi_\alpha \circ f\colon G \to \mathbb{E}.$$

Then F is holomorphic, injective with $F(0) = 0$ and hence $F \in \Phi$. Set $r := |\alpha| < 1$. Then $|\beta| = \sqrt{r}$. An easy calculation using $\psi_\alpha'(z) = \frac{\bar{\alpha}\alpha - 1}{(1-\bar{\alpha}z)^2}$ shows that $|F'(0)| = C|f'(0)|$ with

$$C := |\psi_\beta'(\beta) r_2'(\alpha) \psi_\alpha'(0)| = \frac{r+1}{2\sqrt{r}} > 1.$$

This is a contradiction to the maximality of $|f'(0)|$. $\square$